CONVERGENCE AND DIVERGENCE OF KLEINIAN SURFACE GROUPS

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ABSTRACT. We characterize sequences of Kleinian surface groups with convergent subsequences in terms of the asymptotic behavior of the ending invariants of the associated hyperbolic 3-manifolds. Asymptotic behavior of end invariants in a convergent sequence predicts the parabolic locus of the algebraic limit as well as how the algebraic limit wraps within the geometric limit under the natural locally isometric covering map.

1. INTRODUCTION

Central to Thurston's original approach to the hyperbolization theorem for closed, irreducible, atoroidal 3-manifolds is a collection of compactness criteria for deformation spaces of hyperbolic 3-manifolds. In the Haken setting, such compactness results gave rise to iterative solutions to the search for hyperbolic structures on constituent pieces in a hierarchical decomposition.

Later, the classification of hyperbolic 3-manifolds with finitely generated fundamental group gave explicit *a priori* geometric control of these manifolds in terms of the combinatorics of the asymptotic data determining the hyperbolic structure, up to bi-Lipschitz diffeomorphism. Sullivan's Rigidity Theorem then allows for the passage from bi-Lipschitz diffeomorphism to isometry. The invariants themselves then become parameters, and the bi-Lipschitz control they provide gives rise to a new range of interrelations between geometric and topological features of the resulting manifolds.

The present paper relates these asymptotic invariants explicitly to compactness criteria, characterizing subsequential convergence precisely in terms of the invariants' limiting combinatorics vis a vis the *complex of curves*. In particular, we describe a manner in which invariants *bound projections* to curve complexes of subsurfaces, a notion that guarantees *a priori* bounds for geodesic lengths in a sequence. Our main theorem is a generalization of Thurston's Double Limit Theorem ([38, 35]), which provides a criterion to ensure subsequential convergence of a sequence of Kleinian surface groups, and is a key technical step in Thurston's hyperbolization theorem for 3-manifolds fibering over the circle.

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Theorem 1.1. *Let S be a compact, orientable surface and let* $\{\rho_n\}$ *be a sequence in* $AH(S)$ *with end invariants* $\{\nu_n^\pm\}$*. Then* $\{\rho_n\}$ *has a convergent subsequence if and only if there exists a subsequence* $\{\rho_j\}$ *of* $\{\rho_n\}$ *such that* $\{\nu_j^\pm\}$ *bounds projections.*

We also (see Theorem 1.2) show that the asymptotic behavior of the end invariants predicts the curve and lamination components of the end invariants of the limit and how the algebraic limit manifold "wraps" within a geometric limit.

We briefly describe terms and notation of Theorem 1.1.

Recall that $AH(S)$ is the space of (conjugacy classes of) representations

$$\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$$

for which ρ sends peripheral elements to parabolic elements. The *end invariants* will be discussed more thoroughly in Section 2, but in the case that ρ is *quasi-Fuchsian*, its end invariants $\nu^+(\rho)$ and $\nu^-(\rho)$ are a pair of hyperbolic structures in the Teichmüller space $\mathscr{T}(S)$. In the general setting, each end invariant $\nu^\pm(\rho)$ is a disjoint union of a multicurve on S, the *parabolic locus*, with either an *ending lamination* or a complete finite-area hyperbolic structure supported on each complementary component. A curve c lies in the parabolic locus of $\nu^+(c)$ if it is an upward-pointing parabolic curve, i.e. $\rho(c)$ is parabolic and, after one chooses an orientation-preserving identification of $N_\rho = \mathbb{H}^3/\rho(\pi_1(S))$ with $S \times \mathbb{R}$ in the homotopy class determined by ρ, the cusp of N_ρ associated to c lies in $S \times [r, \infty)$ for some $r \in \mathbb{R}$. Similarly, a curve lies in the parabolic locus of $\nu^-(\rho)$ if and only if it is a downward-pointing parabolic curve.

Given an end invariant ν for ρ and a curve d in $\mathscr{C}(S)$, the *curve complex* of S, we define the *length* $l_\nu(d)$ to be 0 if d is a curve in ν, to be hyperbolic length $l_\tau(d)$ if d lies in a subsurface R admitting a complete hyperbolic structure τ induced by ρ, and to be ∞ otherwise. A collection of non-homotopic essential simple closed curves μ on S is *binding* if any representative of μ on S decomposes S into disks or peripheral annuli. We call a fixed choice of such a collection μ a *coarse basepoint* for $\mathscr{C}(S)$. We define

$$m(\nu,d,\mu) = \max\left\{ \sup_{d\subset\partial Y} d_Y(\nu,\mu), \frac{1}{l_\nu(d)} \right\}$$

where the supremum in the first term is taken over all essential subsurfaces Y with d contained in ∂Y, and the *subsurface projection* $d_Y(\nu,\mu)$ is a measure of the distance in $\mathscr{C}(Y)$ between projections $\pi_Y(\nu)$ and $\pi_Y(\mu)$ to $\mathscr{C}(Y)$ of ν and μ (see sections 2.1 and 2.2).

If we take the supremum of $d_Y(\nu,\mu)$ only over non-annular surface with boundary containing d (i.e. Y is not isotopic to a collar neighborhood $\mathbf{collar}(d)$ of d), then we obtain

$$m^{na}(\nu,d,\mu) = \max\left\{ \sup_{\substack{d\subset\partial Y,\\ Y\neq\mathbf{collar}(d)}} d_Y(\nu,\mu), \frac{1}{l_\nu(d)} \right\}.$$

Choose a coarse basepoint μ in $\mathscr{C}(S)$ once and for all. We say that a sequence $\{\nu_n^{\pm}\}$ of end invariants *bounds projections* if for some $K > 0$ the following conditions hold:

(a) Every geodesic in $\mathscr{C}(S)$ joining $\pi_S(\nu_n^+)$ to $\pi_S(\nu_n^-)$ lies at distance at most K from μ.

(b) If $d \in \mathscr{C}(S)$ is a curve, then either

(i) there exists $\beta(d) \in \{+,-\}$ such that $\{m(\nu_n^{\beta}, d, \mu)\}$ is *eventually bounded*, meaning there is $N \in \mathbb{N}$ such that

$$\sup\{m(\nu_n^{\beta}, d, \mu), n \geq N\} < \infty,$$

or

(ii) $\{m^{na}(\nu_n^+, d, \mu)\}$ and $\{m^{na}(\nu_n^-, d, \mu)\}$ are both eventually bounded and there exists $w(d) \in \mathbb{Z}$ and a sequence $\{s_n\} \subset \mathbb{Z}$ such that $\lim |s_n| = \infty$ and both

$$\{d_Y(D_Y^{s_n w(d)}(\nu_n^+), \mu)\} \quad \text{and} \quad \{d_Y(D_Y^{s_n(w(d)-1)}(\nu_n^-), \mu)\}$$

are eventually bounded when $Y = \mathbf{collar}(d)$ and D_Y is the right Dehn-twist about Y.

In this definition, we say that a curve d is a *combinatorial parabolic* if $\{m(\nu_n^+, d, \mu)\}$ or $\{m(\nu_n^-, d, \mu)\}$ is not eventually bounded. It is an *upward-pointing combinatorial parabolic* if $\{m(\nu_n^+, d, \mu)\}$ is not eventually bounded and $\{m(\nu_n^-, d, \mu)\}$ is eventually bounded. Similarly, we say that a curve d is a *downward-pointing combinatorial parabolic* if $\{m(\nu_n^-, d, \mu)\}$ is not eventually bounded and $\{m(\nu_n^+, d, \mu)\}$ is eventually bounded. We say that d is a *combinatorial wrapped parabolic* if both $\{m(\nu_n^+, d, \mu)\}$ and $\{m(\nu_n^-, d, \mu)\}$ are unbounded. If d is combinatorial parabolic, then we we say that $w(d)$ is its *combinatorial wrapping number*. We notice that all these definitions are independent of the choice of coarse basepoint, so we will usually choose our coarse basepoint to be a complete marking of S (see Section 2.1).

We will see that, for a convergent sequence, every combinatorial parabolic is indeed associated to a parabolic in the limit and furthermore that one can determine which side the parabolic manifests on directly from the asymptotic behavior of $\{m(\nu_n^+, d, \mu)\}$ and $\{m(\nu_n^-, d, \mu)\}$. Moreover, every wrapped parabolic is associated to the wrapping of an immersion of a compact core for N_ρ in a geometric limit of $\{N_{\rho_n}\}$.

We combine our results with [12, Theorem 1.3] to see that the asymptotic behavior of the end invariants predicts the curve and lamination components of the end invariants of the limit.

We also describe, in the case when N_{ρ_n} converges geometrically to a hyperbolic 3-manifold, how a compact core for the algebraic limit is "wrapped" when pushed down into the geometric limit. We describe this phenomenon in terms of a wrapping multicurve and an associated wrapping number (we refer the reader to section 3.1 for definitions). Anderson and Canary [1] first observed that there need not be a compact core for the algebraic limit that embeds in the geometric limit and McMullen [30, Lemma A.4] gave the first description of this phenomenon in the surface group case. We show that there is a compact core for the algebraic limit that embeds in the geometric limit if and only if the wrapping multicurve is empty.

Theorem 1.2. *Suppose that $\{\rho_n\}$ is a sequence in $AH(S)$ converging to $\rho \in AH(S)$ and $\{\nu_n^\pm\}$ bounds projections. Then*

(1) $\ell_\rho(d) = 0$ if and only if d is a combinatorial parabolic for the sequence $\{\nu_n^\pm\}$,

(2) A parabolic curve d is upward-pointing in N_ρ if and only if

$$|m(\nu_n^+, d, \mu)| - |m(\nu_n^-, d, \mu)| \to +\infty.$$

(3) A lamination $\lambda \in \mathcal{EL}(Y)$ is an ending lamination for an upward-pointing (respectively downward-pointing) geometrically infinite end for N_ρ if and only if $\{\pi_Y(\nu_n^+)\}$ (respectively $\{\pi_Y(\nu_n^-)\}$) converges in $\mathcal{C}(Y) \cup \mathcal{EL}(Y)$ to λ.

(4) If $\{\rho_n(\pi_1(S))\}$ converges geometrically to $\hat{\Gamma}$, then the wrapping multicurve for $(\{\rho_n\}, \rho, \hat{\Gamma})$ is the collection of combinatorial wrapping parabolics given by $\{\nu_n^\pm\}$ and if d is a wrapping parabolic, then the combinatorial wrapping number $w(d)$ agree with the actual wrapping number $w^+(d)$.

(5) There is a compact core for N_ρ that embeds in $\hat{N} = \mathbf{H}^3/\Gamma$ if and only if there are no combinatorial wrapping parabolics.

We also obtain the following alternative characterization of convergence in terms of sequence of bounded length multicurves in N_{ρ_n}.

Theorem 1.3. *Let S be a compact, orientable surface and let $\{\rho_n\}$ be a sequence in $AH(S)$. Then $\{\rho_n\}$ has a convergent subsequence if and only if there exists a subsequence $\{\rho_j\}$ of $\{\rho_n\}$ and a sequence $\{c_j^\pm\}$ of pairs of multicurves so that $\{\ell_{\rho_j}(c_j^+ \cup c_j^-)\}$ is bounded and $\{c_j^\pm\}$ bounds projections.*

When c is a multicurve and d is a curve, we define

$$m(c, d, \mu) = \sup_{d \subset \partial Y} d_Y(c, \mu)$$

if $i(c, d) \neq 0$ and $m(c, d, \mu) = \infty$ otherwise. Similarly, we define

$$m^{na}(c, d, \mu) = \sup_{\substack{d \subset \partial Y, \\ Y \neq \mathbf{collar}(d)}} d_Y(c, \mu).$$

In analogy with the end invariants situation, we say that a sequence $\{c_n^\pm\}$ of pairs of multicurves *bounds projections* if, choosing a coarse basepoint μ in $\mathcal{C}(S)$, the following conditions hold:

(a) every geodesic joining $\pi_S(c_n^+)$ to $\pi_S(c_n^-)$ lies a bounded distance from μ in $\mathcal{C}(S)$,

(b) if $d \in \mathcal{C}(S)$ is a curve, then either

(i) there exists $\beta(d)$ such that $\{m(c_n^\beta, d, \mu)\}$ is eventually bounded, or

(ii) $\{m^{na}(c_n^+, d, \mu)\}$ and $\{m^{na}(c_n^-, d, \mu)\}$ are both eventually bounded and there exists $w(d) \in \mathbb{Z}$ and a sequence $\{s_n\} \subset \mathbb{Z}$ such that $\lim |s_n| = \infty$

and both

$$\{d_Y(D_Y^{s_n w(d)}(c_n^+),\mu)\} \quad \text{and} \quad \{d_Y(D_Y^{s_n(w(d)-1)}(c_n^-),\mu)\}$$

are eventually bounded when $Y = \mathbf{collar}(d)$ and D_Y is the right Dehn-twist about Y.

We again say that a curve d is an *upward-pointing combinatorial parabolic* if $\{m(c_n^+,d,\mu)\}$ is not eventually bounded and $\{m(c_n^-,d,\mu)\}$ is eventually bounded. Similarly, we say that a curve d is a *downward-pointing combinatorial parabolic* if $\{m(c_n^-,d,\mu)\}$ is not eventually bounded and $\{m(c_n^+,d,\mu)\}$ is eventually bounded. We say that d is a *combinatorial wrapped parabolic* if both $\{m(c_n^+,d,\mu)\}$ and $\{m(c_n^-,d,\mu)\}$ are unbounded. However, unlike in the end invariant case, the bounded length multicurves bounding projections need not predict the ending laminations or the parabolics in the algebraic limit. For example, if $\{\rho_n\}$ is a convergent sequence, then any constant sequence $\{c_n^\pm\} = \{c^\pm\}$ of pairs of filling multicurves will bound projections. We will discuss this issue further in section 6.

Hausdorff limits of end invariants. We note that Theorems 3–6 and 12 of Ohshika [34], which discuss matters of convergence and divergence of Kleinian groups in the context of convergence of end invariants in the measure and Hausdorff topology on laminations, are special cases of Theorems 1.1 and 1.2. The failure of any of these more traditional forms of convergence of laminations to predict completely the end invariant of the limit, and in turn the presence of a convergent subsequence, is an essential point of the present discussion. The following examples motivate the need for the use of *subsurface projections* to capture convergence phenomena, both here and in [12].

Example 1.4. We use a variation of a construction of Brock [10, Theorem 7.1] to produce sequences $\{\rho_n^1\}$ and $\{\rho_n^2\}$ in $AH(S)$, so that the ending invariants of $\{\rho_n^1\}$ and $\{\rho_n^2\}$ have the same Hausdorff limit and $\{\rho_n^1\}$ and $\{\rho_n^2\}$ have convergent subsequences with algebraic limits whose parabolic loci differ. We further construct sequences $\{\rho_n^3\}$ and $\{\rho_n^4\}$ in $AH(S)$ so that the ending invariants of $\{\rho_n^3\}$ and $\{\rho_n^4\}$ have the same Hausdorff limit, and $\{\rho_n^3\}$ has a convergent subsequence, but $\{\rho_n^4\}$ does not have a convergent subsequence.

We first choose a non-separating curve α on S and a mapping class ψ which restricts to a pseudo-Anosov diffeomorphism of $S - \mathbf{collar}(\alpha)$. We then choose a non peripheral curve γ in $S - \mathbf{collar}(\alpha)$ and a pants decomposition c_0^1 of S, such that all curves in c_0^1 cross α. Let $c_n^1 = D_\gamma^n \circ \psi^n(c_0^1)$ where D_γ is a Dehn-twist about γ. Adjusting if necessary by Dehn twists $D_\alpha^{k_n}$ for suitable powers k_n, the multicurves $\{c_n^1\}$ converge to a Hausdorff limit λ_H which contains γ and intersects α transversely. The lamination λ_H spirals about γ and gives a decomposition of $S \setminus \gamma$ into ideal polygons. One can check that $\{m(c_n^1,d,\mu)\}$ is bounded if d is not either α or γ, and that $m^{na}(c_n^1,\alpha,\mu) \to \infty$ and $m(c_n^1,\gamma,\mu) \to \infty$.

Since λ_H is a limit of multicurves and gives a decomposition of $S \setminus \alpha$ into ideal polygons, one can find a pants decomposition c_0^2 of S such that $\{c_n^2 = D_\gamma^n(c_0^2)\}$

converges to λ_H. One can check that $\{m(c_n^2,d,\mu)\}$ is bounded if d is not γ and that $m(c_n^2,\gamma,\mu)\to\infty$.

Let a be a pants decomposition of S which crosses both α and γ. Let ρ_n^1 have top ending invariant c_n^1 and bottom end invariant a, while ρ_n^2 has top end invariant c_n^2 and bottom end invariant a. The Hausdorff limit of the top ending invariants of both $\{\rho_n^1\}$ and $\{\rho_n^2\}$ is λ_H, while the Hausdorff limit of the bottom ending invariants of each sequence is a. Theorem 1.1 implies that both $\{\rho_n^1\}$ and $\{\rho_n^2\}$ have convergent subsequences. Theorem 1.2 implies that if ρ_∞^1 is the algebraic limit of any convergent subsequence of $\{\rho_n^1\}$, then the upward-pointing parabolic locus of ρ_∞^1 is $\alpha\cup\gamma$, while the downward-pointing parabolic locus is a. On the other hand, if ρ_∞^2 is the algebraic limit of any convergent subsequence of $\{\rho_n^2\}$, then the upward-pointing parabolic locus of ρ_∞^2 is γ, while the downward-pointing parabolic locus is a.

Let b be a pants decomposition of S which crosses γ and contains α. Let ρ_n^3 have top ending invariant c_n^2 and bottom end invariant b, while ρ_n^4 has top ending invariant c_n^1 and bottom end invariant b. The Hausdorff limit of the top ending invariants of both $\{\rho_n^3\}$ and $\{\rho_n^4\}$ is λ_H, while the Hausdorff limit of the bottom end invariants of each sequence is b. Theorem 1.1 implies that $\{\rho_n^3\}$ has a convergent subsequence, but that $\{\rho_n^4\}$ does not have a convergent subsequence.

Example 1.5. If one regards the Hausdorff limit of the end invariants of a sequence of quasifuchsian groups as the Hausdorff limit of a sequence of minimal length pants decompositions in the associated conformal structures, as Ohshika [34] does, then one may use the wrapping construction to construct simpler examples.

Let α be a non-peripheral curve on S. Let X be a hyperbolic surface with unique minimal length pants decomposition r which crosses α. Let τ_n^1 be a quasifuchsian group with top end invariant $D_\alpha^{3n}(X)$ and bottom end invariant $D_\alpha^{2n}(X)$. The Hausdorff limit of the top and bottom end invariants of $\{\tau_n^1\}$ is the lamination λ obtained by "spinning" r about α. Theorem 1.1 implies that $\{\tau_n^1\}$ has a convergent subsequence, while Theorem 1.2 implies that if τ_∞^1 is the algebraic limit of any convergent subsequence of $\{\tau_n^1\}$, then the upward-pointing parabolic locus of τ_∞^1 is α, while the downward pointing parabolic locus is empty.

Let τ_n^2 be a quasifuchsian group with top end invariant $D_\alpha^{n}(X)$ and bottom end invariant $D_\alpha^{2n}(X)$. The Hausdorff limit of the top and bottom end invariants of $\{\tau_n^2\}$ is again λ. Theorem 1.1 implies that $\{\tau_n^2\}$ has a convergent subsequence, while Theorem 1.2 implies that if τ_∞^2 is the algebraic limit of any convergent subsequence of $\{\tau_n^2\}$, then the upward-pointing parabolic locus of τ_∞^2 is empty, while the downward pointing parabolic locus is α.

Let τ_n^3 be a quasifuchsian group with top end invariant $D_\alpha^{2n}(X)$ and bottom end invariant $D_\alpha^{2n}(X)$. The Hausdorff limit of the top and bottom end invariants of $\{\tau_n^3\}$ is again λ. Theorem 1.1 implies that $\{\tau_n^3\}$ has no convergent subsequences.

Outline of the paper: In section 2 we recall definitions and previous results that will be used in the paper. In section 3 we define the wrapping multicurve and the wrapping numbers. We assume that $\{\rho_n\}$ converges to ρ and that $\{N_{\rho_n}\}$ converges geometrically to $\hat{N}$. Let $\pi:N_\rho\to\hat{N}$ be the obvious covering map. We first find a level surface F in N_ρ and a collection Q of incompressible annuli in F, so that $\pi|_F$ is an immersion, $\pi|_{F-Q}$ is an embedding and π wraps Q around the boundary of a cusp region in $\hat{N}$. The collection q of core curves of elements of Q is the wrapping multicurve. The wrapping number then records "how many times" Q is wrapped around the cusp region.

In section 4, we prove that if a sequence $\{\rho_n\} \subset AH(S)$ converges, then some subsequence of its end invariants predicts convergence. We also establish Theorem 1.2. We first use work of Minsky [31, 32] and Brock-Bromberg-Canary-Minsky [12] to establish the results in the case that the wrapping multicurve is empty. When the wrapping multicurve is non-empty, we use the wrapped surface F from section 3.1 to construct two new sequences, that differ from the original sequence by powers of Dehn twists in components of the wrapping multicurve, but themselves have empty wrapping multicurves. We can then apply the results from the empty wrapping multicurve case to both of these sequences. Analyzing the relationship between the end invariants of the original sequence and the two new sequences allow us to complete the proof.

In section 5, we show that if the sequence $\{\nu_n^\pm\}$ of end invariants for a sequence $\{\rho_n\}$ in $AH(S)$ bounds projections, then one can find a subsequence $\{\rho_j\}$ and a sequence $\{c_j^\pm\}$ of pairs of multicurves such that $\{\ell_{\rho_j}(c_j^+ \cup c_j^-)\}$ is bounded and $\{c_j^\pm\}$ bounds projections. The difficulty comes from the fact that one must insure that c_n^+ and c_n^- do not share any curves while bounding projections. In particular one must take special care of the curves where $\{m(\nu_n^\beta, d, \mu)\}$ is unbounded. To overcome these difficulties, we will construct $c_n^\pm$ as minimal length pants decompositions under some constraints.

In section 6, we show that if $\{\rho_n\}$ is a sequence in $AH(S)$ and there is a sequence of bounded length multicurves $\{c_n^\pm\}$ that bound projections, then $\{\rho_n\}$ has a convergent subsequence. Again we start with the case that the wrapping multicurve is empty. We may assume that each $c_n^\pm$ is a pants decomposition of S. We first use results of Minsky [31] to find a pants decomposition r such that $\{\ell_{\rho_n}(r)\}$ is bounded. We then construct the model manifold M_n^β associated to the hierarchy joining r to c_n^β and observe, using work of Bowditch [9] and Minsky [32], that there is a uniformly Lipschitz map of M_n^β into N_{ρ_n}. (If r and c_n^β share curves we consider a model manifold associated to a subsurface of S.) We find a bounded length transversal in M_n to each curve in r and then observe that it also has bounded length in N_{ρ_n}. We pass to a subsequence so that the sequence of transversals we have constructed is constant and then simply apply the Double Limit Theorem to conclude that there is a convergent subsequence. When the wrapping multicurve is not empty, we construct two new sequences with empty wrapping multicurves and use them to produce a converging subsequence of the original sequence.

Finally, in section 7 we combine the results of sections 4, 5 and 6 to complete the proofs of both Theorems 1.1 and 1.3.

2. Background

In this section, we collect definitions and previous results which will be used in the paper. We first need to recall the definitions of curve complexes of subsurfaces, subsurface projections, markings and end invariants.

2.1. **Curve complexes, markings and subsurface projections.** If W is an essential non-annular subsurface of S, its curve complex $\mathscr{C}(W)$ is a locally infinite simplicial complex whose vertices are isotopy classes of essential non-peripheral curves on W. Two vertices are joined by an edge if and only if the associated curves intersect minimally. A collection of $n+1$ vertices span a n-simplex if the corresponding curves have mutally disjoint representatives. Masur and Minsky [27] proved that $\mathscr{C}(W)$ is Gromov hyperbolic with respect to its natural path metric.

We will assume throughout that all curves are essential and non-peripheral. A *multicurve* will be a collection of disjoint curves, no two of which are homotopic. A *pants decomposition* of W is a maximal multicurve.

Klarreich [23], see also Hamenstadt [19], showed that the Gromov boundary $\partial_\infty\mathscr{C}(W)$ of $\mathscr{C}(W)$ can be naturally identified with the space $\mathscr{E}\mathscr{L}(W)$ of filling geodesic laminations on W.

A *marking* μ on S is a multicurve base(μ) together with a selection of transversal curves, at most one for each component of base(μ). A transversal curve to a curve c in base(μ) intersects c and is disjoint from base$(\mu)-c$. A marking is *complete* if base(μ) is a pants decomposition and every curve in base(μ) has a transversal. A *generalized marking* is a collection of filling laminations on a disjoint collection of subsurfaces together with the boundary of those subsurfaces and a marking of their complement. (See Masur-Minsky [28] and Minsky [32] for a more careful discussion of markings and generalized markings.)

If W is an essential non-annular subsurface, one may define a *subsurface projection*

$$\pi_W : \mathscr{C}(S) \to \mathscr{C}(W) \cup \{\emptyset\}.$$

If $c \in \mathscr{C}(S)$ and c is disjoint from W, then $\pi_W(c) = \emptyset$. If not, $c \cap W$ is a collection of arcs and curves on W. Each arc in $c \cap W$ may be surgered to produce an essential curve on W by adding arcs in ∂W. We let $\pi_W(c)$ denote a choice of one of the resulting essential curves in W; then $\pi_W(c)$ is *coarsely* well-defined - any two choices lie at bounded distance (see [28, Lemma 2.3]). For a subset μ of $\mathscr{C}(S)$ (such as a multicurve, a marking or a coarse basepoint for $\mathscr{C}(S)$), we choose $\pi_W(\mu)$ to be a curve in $\bigcup_{c\in\mu} \pi_W(c)$ if there is one and to be $\emptyset$ otherwise. We can then define

$$d_W(c,\mu) = d_{\mathscr{C}(W)}(\pi_W(c), \pi_W(\mu))$$

if $\pi_W(c) \neq \emptyset$ and $\pi_W(\mu) \neq \emptyset$, and define $d_W(c,\mu) = +\infty$ otherwise.

If μ is a generalized marking on S, then we define

$$\pi_W(\mu) \in \mathscr{C}(W) \cup \mathscr{E}\mathscr{L}(W) \cup \emptyset$$

by

(1) letting $\pi_W(\mu) = \emptyset$ if μ does not intersect W,
(2) letting $\pi_W(\mu) = \lambda$ if $\lambda \subset \mu$ lies in $\mathscr{E}\mathscr{L}(W)$,

(3) constructing $\pi_W(\mu)$ as above using any simple closed curve or proper arc in $\mu \cap W$.

For a pair of generalized markings, we define

$$d_W(\mu,\mu') = d_{\mathscr{C}(W)}(\pi_W(\mu),\pi_W(\mu'))$$

if $\pi_W(\mu),\pi_W(\mu') \in \mathscr{C}(W)$ and $d_W(\mu',\mu) = \infty$ if $\pi_W(\mu)$ or $\pi_W(\mu')$ lies in $\mathscr{E}\mathscr{L}(W) \cup \{\emptyset\}$.

If W is an essential annulus in S we may also define $d_W(c,d)$ and $d_W(c,\mu)$. The simplest way to do this is to first fix a hyperbolic metric on S and let $\tilde{S}$ be the annular cover S so that W lifts to a compact core for $\tilde{S}$. We then compactify $\tilde{S}$ by its ideal boundary to obtain an annulus A and define a complex $\mathscr{C}(W)$ whose vertices are geodesics in A that joins the two boundary components of A. We join two vertices if they have disjoint representatives. If we give $\mathscr{C}(W)$ the natural path metric then $d_{\mathscr{C}(W)}(a,b) = i(a,b)+1$ and it follows that $\mathscr{C}(W)$ is quasi-isometric to $\mathbb{Z}$. Given a simple closed curve $c \subset S$, we realize it as geodesic and then consider its pre-image in $\tilde{S}$. If c intersects W essentially, the pre-image contains an essential arc $\tilde{c}$ whose closure joins the two boundary components of A, we set $\pi_W(c) = \tilde{c}$ and we set $\pi_W(c) = \emptyset$ otherwise. For a subset μ of $\mathscr{C}(S)$, we again choose $\pi_W(\mu)$ to be an element of $\bigcup_{c\in\mu} \pi_W(c)$ if there is one and to be $\emptyset$ otherwise. We can then define

$$d_W(c,\mu) = d_{\mathscr{C}(W)}(\pi_W(c),\pi_W(\mu))$$

if $\pi_W(c) \neq \emptyset$ and $\pi_W(\mu) \neq \emptyset$, and define $d_W(c,\mu) = +\infty$ otherwise. One can check that this definition is independent of the choice of metric. (Again see Masur-Minsky [28] and Minsky [32] for a complete discussion of subsurface projections and the resulting distances.)

In all cases, the distance between two curves (or markings) is bounded above by a function of their intersection number.

Lemma 2.1. ([27, Lemma 2.1]) *If S is a compact orientable surface, α,β are multicurves or markings on S and W is an essential subsurface of S, then*

$$d_W(\alpha,\beta) \leq 2i(\alpha,\beta)+1.$$

The following estimate is often useful in establishing relationships between subsurface projections. Behrstock ([5, Theorem 4.3]) first gave a version with inexplicit constants which depends on the surface S. We will use a version, due to Leininger, with explicit universal constants.

Lemma 2.2. ([25, Lemma 2.13]) *Given a compact surface S, two essential subsurfaces Y and Z which overlap and a generalized marking μ which intersects both Y and Z, then*

$$d_Y(\mu,\partial Z) \geq 10 \Longrightarrow d_Z(\mu,\partial Y) \leq 4$$

We will also use the fact that a sequence of curves which is not eventually constant blows up on some subsurface.

Lemma 2.3. *Given a sequence of simple closed curves $\{c_n\}$ and a complete marking μ on a compact surface S, there is a subsequence $\{c_j\}$ such that either $\{c_j\}$ is constant or there is a subsurface $Y \subseteq S$ with $d_Y(\mu,c_j) \longrightarrow \infty$.*

Proof. Fix a metric on S and realize the sequence $\{c_n\}$ as a sequence of closed geodesics. We then extract a subsequence $\{c_j\}$ that converges in the Hausdorff topology on closed subsets of S to a geodesic lamination λ. If λ contains an isolated simple closed curve, then $\{c_j\}$ is eventually constant and we are done. If not, let Y be the supporting subsurface of a minimal sublamination λ_0 of λ. If Y is not an annulus, then $\lambda_0 \in \mathcal{EL}(S)$ and results of Klarreich [23, Theorem 1.4] (see also Hamenstadt [19]) imply that $d_Y(\mu, c_j) \to \infty$.

If λ_0 is a simple closed geodesic, then $Y = \mathbf{collar}(\lambda_0)$ is an annulus and, since λ doesn't contain an isolated simple close curve, there must be leaves of λ spiraling around λ_0. Let $\tilde{S}_0$ be the annular cover of S associated to the cyclic subgroup of $\pi_1(S)$ generated by λ_0 and let $\tilde{\lambda}_0$ be the unique lift of λ_0 to $\tilde{S}_0$. Let $\tilde{c}_j = \pi_Y(c_j)$. Since $\{c_j\}$ converges to λ in the Hausdorff topology and there exist leaves of λ spiraling about λ_0, the acute angle between $\tilde{c}_j$ and $\tilde{\lambda}_0$ converges to 0. It follows that $i(\tilde{c}_j, \tilde{a}) \to \infty$ for any fixed element $\tilde{a} \in \mathcal{C}(Y)$. In particular, if d is a component of μ that intersects λ_0 and $\tilde{d} = \pi_Y(d)$, then

$$d_Y(c_j, d) = d_{\mathcal{C}(Y)}(\tilde{c}_j, \tilde{d}) = i(\tilde{c}_j, \tilde{d}) + 1 \to \infty.$$

It follows that $d_Y(c_j, \mu) \to \infty$ as desired. □

2.2. **End invariants.** If $\rho \in AH(S)$, the end invariants of N_ρ encode the asymptotic geometry of $N_\rho = \mathbf{H}^3/\rho(\pi_1(S))$. The Ending Lamination Theorem (see Minsky [32] and Brock-Canary-Minsky [13]) asserts that a representation $\rho \in AH(S)$ is uniquely determined by its end invariants. The reader will find a more extensive discussion of the definition of the end invariants and the Ending Lamination Theorem in Minsky [32].

A $\rho(\pi_1(S))$-invariant collection $\mathcal{H}$ of disjoint horoballs in $\mathbf{H}^3$ is a *precisely invariant collection of horoballs* for $\rho(\pi_1(S))$ if there is a horoball based at the fixed point of every parabolic element of $\rho(\pi_1(S))$ (and every horoball in $\mathcal{H}$ is based at a parabolic fixed point). The existence of such a collection is a classical consequence of the Margulis Lemma, see [26, Proposition VI.A.11] for example. We define

$$N_\rho^0 = (\mathbf{H}^3 - \bigcup_{H \in \mathcal{H}} H)/\rho(\pi_1(S)).$$

If $\mathcal{H}_p$ denotes the set of horoballs in $\mathcal{H}$ which are associated to peripheral elements of $\pi_1(S)$, then we define

$$N_\rho^1 = (\mathbf{H}^3 - \bigcup_{H \in \mathcal{H}_p} H)/\rho(\pi_1(S)).$$

A *relative compact core* for N_ρ^0 is a compact submanifold M_ρ of N_ρ^0 such that the inclusion of M_ρ into N_ρ is a homotopy equivalence and M_ρ intersects each component of ∂N_ρ^0 in an incompressible annulus. Let $P_\rho = M_\rho \cap \partial N_\rho^0$ and let $P_\rho^1 = M_\rho \cap \partial N_\rho^1$. (See Kulkarni-Shalen [24] and McCullough [29] for proofs that N_ρ^0 admits a relative compact core.)

Bonahon [8] showed that there is an orientation preserving homeomorphism from $S \times \mathbb{R}$ to N_ρ^1 in the homotopy class determined by ρ. We will implicitly identify N^1_ρ with $S\times\mathbb{R}$ throughout the paper. Suppose that W is a subsurface of S and $f : W \to N^1_\rho$ is a map of W into $S\times\mathbb{R}$ (in the homotopy class associated to $\rho|_{\pi_1(W)}$). We say that f (or $f(W)$) is a *level subsurface* if it is an embedding which is isotopic to $W\times\{0\}$. If $W = S$, we say f (or $f(S)$) is a *level surface*.

The *conformal boundary* $\partial_c N_\rho$ of N_ρ is the quotient by Γ of the domain $\Omega(\rho)$ of discontinuity for the action of $\rho(\Gamma)$ on $\hat{\mathbb{C}}$. One may identify the conformal boundary $\partial_c N_\rho$ with a collection of components of $\partial M_\rho - P_\rho$. The other components of $\partial M_\rho - P_\rho$ bound neighborhoods of geometrically infinite ends of N^0_ρ. If E is a geometrically infinite end with a neighborhood bounded by a component W of $\partial M_\rho - P_\rho$, then there exists a sequence $\{\alpha_n\}\subset\mathscr{C}(W,\rho,L_1)$, for some $L_1 = L_1(S) > 0$, whose geodesic representatives $\{\alpha_n^*\}$ exit E (see Lemma 2.9 for a more careful statement). The sequence $\{\alpha_n\}$ converges to an ending lamination $\lambda\in\mathscr{E}\mathscr{L}(S)$ and we call λ the *ending lamination* of E (λ does not depend on the choice of the sequence $\{\alpha_n\}$). Moreover, if $\{\beta_n\}$ is any sequence in $\mathscr{C}(W)$ which converges to λ, then the sequence $\{\beta_n^*\}$ of geodesic representatives in N_ρ exits E. (See Bonahon [8] for an extensive discussion of geometrically infinite ends.)

There exists an orientation-preserving homeomorphism of $S\times I$ with M_ρ, again in the homotopy class determined by ρ, so that $\partial S\times I$ is identified with P^1_ρ. Let P^+_ρ denote the components of P_ρ contained in $S\times\{1\}$ and let P^-_ρ denote the component of P_ρ contained in $S\times\{0\}$. A core curve of a component of P^+_ρ is called an *upward-pointing parabolic curve* and a core curve of a component of P^-_ρ is called a *downward-pointing parabolic curve*. Similarly, a component of $\partial_c N_\rho$ or a geometrically infinite end of N_ρ is called *upward-pointing* if it is identified with a a subsurface of $S\times\{1\}$, and is called *downward-pointing* if it is identified with a subset of $S\times\{0\}$.

The *end invariant* ν^+_ρ consists of the multicurve p^+ of upward-pointing parabolic curves together with a conformal structure on each geometrically finite component of $S\times\{1\}-p^+$, coming from the conformal structure on the associated component of the conformal boundary, and a filling lamination on each geometrically infinite component, which is the ending lamination of the associated end. The end invariant ν^-_ρ is defined similarly.

If ν is an end invariant, we define an associated generalized marking $\mu(\nu)$. We let base$(\mu(\nu))$ consist of all the curve and lamination components of ν together with a minimal length pants decomposition of the conformal (hyperbolic) structure on each geometrically finite component. For each curve in the minimal length pants decomposition of a geometrically finite component we choose a minimal length transversal. Notice that the associated marking is well-defined up to uniformly bounded ambiguity.

Given $\rho\in AH(S)$ with end invariants $\nu^\pm$, we then define, for each essential subsurface W of S,

$$\pi_W(\nu^\pm) = \pi_W(\mu(\nu^\pm)).$$

Property (3) in Theorem 1.2 can be viewed as a continuity property for the projections of end invariants to subsurfaces. This property was established by Brock-Bromberg-Canary-Minsky in [12]:

Theorem 2.4. ([12, Theorem 1.1]) *Let $\rho_n \longrightarrow \rho$ in $AH(S)$. If $W \subseteq S$ is an essential subsurface of S, other than an annulus or a pair of pants, and $\lambda \in EL(W)$ is a lamination supported on W, then the following statements are equivalent :*
(1) λ is a component of ν_ρ^+.
(2) $\{\pi_W(\nu_{\rho_n}^+)\}$ converges to λ.

2.3. **The bounded length curve set.** The Ending Lamination Theorem [32, 13] assures that the end invariants coarsely determine the geometry of N_ρ. In particular, one can use the end invariants to bound the lengths of curves in N_ρ and to coarsely determine the set of curves of bounded length. We will need several manifestations of this principle.

It is often useful to, given $L > 0$, consider the set of all curves in N_ρ with length at most L. We define

$$\mathscr{C}(\rho,L) = \{d \in \mathscr{C}(S) \mid \ell_\rho(d) \leq L\}.$$

Minsky, in [31], showed that if the projection of $\mathscr{C}(\rho,L)$ to $\mathscr{C}(W)$ has large diameter, then ∂W is short in N_ρ.

Theorem 2.5. ([31, Theorem 2.5]) *Given S, $\varepsilon > 0$ and $L > 0$, there exists $B(\varepsilon,L)$ such that if $\rho \in AH(S)$, $W \subset S$ is a proper subsurface and*

$$\mathrm{diam}(\pi_W(\mathscr{C}(\rho,L))) > B(\varepsilon,L),$$

then $l_\tau(\partial W) < \varepsilon$.

In [12] it is proven that $\pi_W(\mathscr{C}(\rho,L))$ is well-approximated by a geodesic joining $\pi_W(\nu^+)$ to $\pi_W(\nu^-)$.

Theorem 2.6. ([12, Theorem 1.2]) *Given S, there exists $L_0 > 0$ such that for all $L \geq L_0$, there exists $D_0 = D_0(L)$, such that, if $\rho \in AH(S)$ has end invariants $\nu^\pm$, and $W \subset S$ is an essential subsurface more complicated than a thrice-punctured sphere, then $\pi_W(C(\rho,L))$ has Hausdorff distance at most D_0 from any geodesic in $\mathscr{C}(W)$ joining $\pi_W(\nu^+)$ to $\pi_W(\nu^-)$. Moreover, if $d_W(\nu^+,\nu^-) > D_0$, then*

$$C(W,\rho,L) = \{\alpha \in \mathscr{C}(W) : l_\alpha(\rho) < L\}$$

is nonempty and also has Hausdorff distance at most D_0 from any geodesic in $\mathscr{C}(W)$ joining $\pi_W(\nu^+)$ to $\pi_W(\nu^-)$.

As a generalization of Minsky's a priori bounds (see [32, Lemma 7.9]), Bowditch proved that all curves on a tight geodesic in $\mathscr{C}(W)$ joining two bounded length multicurves, also have bounded length. We recall that if W is a non-annular essential subsurface of S, then a *tight geodesic* is a sequence $\{w_i\}$ of simplices in $\mathscr{C}(W)$ such that if v_i is a vertex of w_i and v_j is a vertex of w_j, then $d_W(v_i,v_j) = |i-j|$ and each w_i is the boundary of the subsurface filled by $w_{i-1} \cup w_{i+1}$.

Theorem 2.7. (Bowditch [9, Theorem 1.3]) *Let S be a compact orientable surface. Given $L > 0$ there exists $R(L,S)$ such that if $\rho \in AH(S)$, W is an essential non-annular subsurface of S, $\{w_i\}_{i=0}^n$ is a tight geodesic in $\mathscr{C}(W)$, and $\ell_\rho(w_0) \leq L$ and $\ell_\rho(w_n) \leq L$, then*

$$\ell_\rho(w_i) \leq R$$

for all $i = 1,\dots,n-1$.

2.4. **Margulis regions and topological ordering.** There exists a constant $\epsilon_3 > 0$, known as the *Margulis constant*, such that if $\epsilon \in (0,\epsilon_3)$ and N is a hyperbolic 3-manifold, then each component of the *thin part*

$$N_{thin(\epsilon)} = \{x \in N \mid \mathrm{inj}_N(x) < \epsilon\}$$

is either a solid torus neighborhood of a closed geodesic or the quotient of a horoball by a group of parabolic elements (see [37, Corollary 5.10.2] for example). If $\rho \in AH(S)$ and d is a curve on S, then let $\mathbb{T}_\epsilon(d)$ be the component of $N_{thin(\epsilon)}$ whose fundamental group is generated by d. With this definition, $\mathbb{T}_\epsilon(d)$ will often be empty. When it is non-empty, we will call it a *Margulis region* and when it is non-compact we will call it a *Margulis cusp region*. Notice that if $N = \mathbf{H}^3/\Gamma$, then the pre-image in $\mathbf{H}^3$ of all the non-compact components of $N_{thin(\epsilon)}$, for any $\epsilon \in (0,\epsilon_3)$, is a precisely invariant system of horoballs for Γ.

Suppose that α and β are homotopically non-trivial curves in N^1_ρ and that their projections to S intersect essentially. We say that α lies above β if α may be homotoped to $+\infty$ in the complement of α (i.e. α may be homotoped into $S\times[R,\infty)$ in the complement of β for all R). Similarly, we say that β is *below* α if β may be homotoped to $-\infty$ in the complement of α (see [12, §2.5] for a more detailed discussion).

It is shown in [12] that if the geodesic representative of a curve d lies above the geodesic representative of the boundary component of a subsurface W, then the projection of d lies near the projection of ν^+.

Theorem 2.8. ([12, Theorem 1.3]) *Given S and $L > 0$ there exists $D = D(S,L)$ such that if $\alpha \in \mathscr{C}(S)$, $\rho \in AH(S)$ has end invariants $\nu^\pm$, $l_\rho(\alpha) < L$, α overlaps a proper subsurface $W \subset S$ (other than a thrice-punctured sphere), and there exists a component β of ∂W such that α^* lies above β^* in N_ρ, then*

$$d_W(\alpha,\nu^+) < D.$$

Remark: If $\rho(\alpha)$ is parabolic, then α has no geodesic representative in N_ρ. If α is an upward-pointing parabolic, it is natural to say that it lies above the geodesic representative of every curve it overlaps, while if α is a downward-pointing parabolic, it is natural to say that it lies below the geodesic representative of every curve it overlaps.

The following observation is a consequence of the geometric description of geometrically infinite ends (see Bonahon [8]).

Lemma 2.9. *Given a compact surface S, there exists $L_1 = L_1(S)$ such that if $\rho \in AH(S)$, W is an essential sub-surface of S which is the support of a geometrically infinite end E of N^0_ρ and Δ is a finite subset of $\mathscr{C}(W)$, then there exists a pants decomposition r of W such that $l_\rho(r) \le L_1$ and any curve in r lies above, respectively below, any curve in Δ when E is upward-pointing, respectively downward-pointing.*

2.5. **Lipschitz surfaces and bounded length curves.** If $\rho \in AH(S)$, then a *K-Lipschitz surface* in N_ρ is a π_1-injective K-Lipschitz map $f : X \to N_\rho$ where X is a (complete) finite area hyperbolic surface. Incompressible pleated surfaces, see Thurston [37, section 8.8] and Canary-Epstein-Green [16, Chapter I.5], are examples of 1-Lipschitz surfaces. If W is an essential subsurface of S and $\alpha \in \mathscr{C}(W)$, then we say that a K-Lipschitz surface $f : X \to N_\rho$, where X is a hyperbolic structure on $\text{int}(W)$, *realizes* the pair (α, W) if there exists a homeomorphism $h : \text{int}(W) \to X$ such that $(f \circ h)_*$ is conjugate to $\rho|_{\pi_1(W)}$ and $f(h(\alpha)) = \alpha^*$. Thurston observed that if $\rho(\pi_1(\partial W))$ is purely parabolic and $\rho(\alpha)$ is hyperbolic, then one may always find a pleated surface realizing (α, W).

Lemma 2.10. (Thurston [37, Section 8.10], Canary-Epstein-Green [16, Theorem I.5.3.6]) *Suppose that $\rho \in AH(S)$, W is an essential subsurface of S and $\alpha \in \mathscr{C}(W)$. If every (non-trivial) element of $\rho(\pi_1(\partial W))$ is parabolic and $\rho(\alpha)$ is hyperbolic, then there exists a 1-Lipschitz surface realizing (α, W).*

One may use Lemma 2.10 and a result of Bers ([7], see also [15, p.123]) to construct bounded length pants decompositions which include any fixed bounded length curve.

Lemma 2.11. *Suppose that $\rho \in AH(S)$, W is an essential subsurface of S and $\alpha \in \mathscr{C}(W)$. Given $L > 0$, there is $L' = L'(L, S)$ such that, if $\ell_\rho(\alpha) + \ell_\rho(\partial W) \le L$, then W admits a pants decomposition p containing α such that $\ell_\rho(p) \le L'$.*

2.6. **Geometric limits.** A sequence $\{\Gamma_n\}$ of Kleinian groups *converges geometrically* to a Kleinian group $\hat{\Gamma}$ if every accumulation point γ of every sequence $\{\gamma_n \in \Gamma_n\}$ lies in $\hat{\Gamma}$ and if every element α of Γ_∞ is the limit of a sequence $\{\alpha_n \in \Gamma_n\}$. It is useful, to think of geometric convergence of a sequence of torsion-free Kleinian groups, in terms of geometric convergence of the sequence of hyperbolic 3-manifolds. The following result combines standard results about geometric convergence which will be used in the paper.

Lemma 2.12. *Suppose that $\{\rho_n : \pi_1(S) \to \text{PSL}(2,\mathbb{C})\}$ is a sequence of discrete faithful representations converging to the discrete faithful representation $\rho : \pi_1(S) \to \text{PSL}(2,\mathbb{C})$. Then, there exists a subsequence $\{\rho_j\}$ so that $\{\rho_j(\pi_1(S))\}$ converges geometrically to $\hat{\Gamma}$.*

Let $\hat{N} = \mathbb{H}^3/\hat{\Gamma}$ and let $\pi : N_\rho \to \hat{N}$ be the natural covering map. Let $\mathscr{H}$ be a precisely invariant system of horoballs for $\hat{\Gamma}$.

There exists a nested sequence $\{Z_j\}$ of compact sub-manifolds exhausting $\hat{N}$ and K_j-bilipschitz smooth embeddings $\psi_j : Z_j \to N_{\rho_j}$ such that:

(1) $K_j \to 1$.
(2) If V is a compact component of $\partial \hat{N}^0$, then, for all large enough n, $\psi_j(\partial V)$ is the boundary of a Margulis region for N_{ρ_j}.
(3) If Q is a compact subset of a non-compact component of $\partial \hat{N}^0$, then, for all large enough j, $\psi_j(Q)$ is contained in the boundary of a Margulis region V_j for N_{ρ_j} and $\psi_j(Z_j \cap \hat{N}^0)$ does not intersect V_j.

(4) If X is a finite complex and $h : X \to N_\rho$ is continuous, then, for all large enough j, $(\psi_j \circ \pi \circ h)_*$ is conjugate to $\rho_j \circ \rho^{-1} \circ h_*$.

Proof. The existence of the subsequence $\{\rho_j\}$ is guaranteed by Canary-Epstein-Green [16, Thm. 3.1.4]. The existence of the sub-manifolds $\{Z_n\}$ and the comparison maps $\{\psi_n\}$ with property (1) is given by [16, Thm. 3.2.9]. Properties (2) and (3) are obtained by Brock-Canary-Minsky [13, Lemma 2.8]. Property (4) is observed in [13, Prop. 2.7], see also Anderson-Canary [2, Lemma 7.2]. □

3. THE WRAPPING MULTICURVE

In this section, we analyze how compact cores for algebraic limits immerse into geometric limits. We will see that if $\{\rho_n\} \subset AH(S)$ converges algebraically to ρ and $\{\rho_n(\pi_1(S))\}$ converges geometrically to $\hat{\Gamma}$, then there is a level surface $F \subset N^0_\rho$ and a collection Q of incompressible annuli in F so that the covering map $\pi : N^0_\rho \to \hat{N}^0$ is an embedding on $F - Q$ and (non-trivially) wraps each component of Q around a toroidal component of $\partial \hat{N}^0_\rho$. The collection q of core curves of Q is called the wrapping multicurve and we will define a wrapping number associated to each component of q which records how many times the surface wraps the associated annulus around the toroidal component of $\partial \hat{N}^0_\rho$.

3.1. **Wrapped surfaces.** We first examine the topology of the situation. Given a compact non-annular surface G and $e \in \mathscr{C}(G)$, let $E = \textbf{collar}(e)$ be an open collar neighborhood of e on G, $\hat{G} = G - E$,

$$X = G \times [-1,1] \quad \text{and} \quad \hat{X} = X - V \quad \text{where} \quad V = E \times (-\frac{1}{2}, \frac{1}{2}) \subset X$$

is a solid torus in the homotopy class of e. If $T = \partial V$ and $\hat{Z} = \hat{G} \times \{0\} \cup T$, then $\hat{Z}$ is a spine for $\hat{X}$. An orientation on G determines an orientation on X and hence on V which induces an orientation on T. Let m be an essential curve on T that bounds a disk in V and let l be one of the components of $\partial \bar{E} \times \{0\}$. We orient this meridian and longitude so that the orientation of (m,l) agrees with the orientation of T. We also decompose T into two annuli with

$$A = \partial \bar{E} \times [0,1/2] \cup E \times \{1/2\} \quad \text{and} \quad B = \partial \bar{E} \times [-1/2,0] \cup E \times \{-1/2\}.$$

We will show that every map from G to $\hat{X}$ that is homotopic, in X, to a level inclusion, is homotopic, in X to exactly one of a family $\{f_k : G \to \hat{X}\}_{k \in \mathbb{Z}}$ of standard wrapping maps. Let $f_1 : G \to X$ be an embedding such that the restriction of f_1 to $\hat{G}$ is $id \times \{0\}$, i.e. $f_1(x) = (x,0)$ if $x \in \hat{G}$, and $f_1|_{\bar{E}}$ is a homeomorphism to A. For all $k \in \mathbb{Z}$, let $\phi_k : T \to T$ be an immersion which is the identity on B and wraps A "k times around" T, namely $(\phi_k)_*(m) = km$ and $(\phi_k)_*(l) = l$. We then define $f_k : G \to \hat{Z} \subset X$ by $f_k|_{\hat{G}} = f_1|_{\hat{G}}$ and $f_k|_{\bar{E}} = \phi_k \circ f_1|_{\bar{E}}$. Note that all of these maps are homotopic as maps to X. As maps to $\hat{Z}$ (or $\hat{X}$) they are homotopically distinct as can be seen by counting the algebraic intersection with a point on A and a point on B. We will call k the *wrapping number* of f_k.

The next lemma allows us to define a wrapping number for any map in the correct homotopy class.

Lemma 3.1. *Let $g : (G, \partial G) \to (\hat{X}, \partial G \times [-1,1])$ be a map such that g is homotopic to $id \times \{0\}$, as a map into $(X, \partial G \times [-1,1])$. Then there exists a unique $k \in \mathbb{Z}$ such that g is homotopic to f_k as a map into $(\hat{X}, \partial G \times [-1,1])$.*

Proof. Since $\hat{Z}$ is a spine of $\hat{X}$, we may assume that the image of g lies in $\hat{Z}$. Since g is homotopic to the level inclusion $id \times \{0\}$ on $\hat{G}$, we may homotope g within $\hat{X}$ so that g agrees with the level inclusion on $\hat{G}$. Since every immersed incompressible annulus in $\hat{X}$ with boundary in T is homotopic, rel boundary, into T, we can further homotope g, rel $\hat{G}$, such that $g(E) \subset T$. A simple exercise shows that any map of E to T that agrees with $id \times \{0\}$ on ∂E is homotopic to the composition of ϕ_k, for some k, and some power of a Dehn twist about E. Since g is homotopic to $id \times \{0\}$ within X, the Dehn twist is un-necessary, so g is homotopic to f_k in $(\hat{X}, \partial G \times [-1,1])$. □

We recall that we will be considering the case where $\{\rho_n\} \subset AH(S)$ converges to ρ, $\{\rho_n(\pi_1(S))\}$ converges to $\hat{\Gamma}$, and there is a level surface $F \subset N^0_\rho$ and a collection E of incompressible annuli in F so that the covering map $\pi : N^0_\rho \to \hat{N}^0$ is an embedding on $F - E$ and (non-trivially) wraps each component of E around a toroidal component of $\partial \hat{N}^0_\rho$. We also have, for large enough n, a 2-bilipschitz map $\psi_n : \hat{N} \to N_{\rho_n}$ defined on a regular neighborhood of $\pi(F)$ so that each component of $\psi_n(\pi(E))$ bounds a Margulis tube in N_{ρ_n}. The following lemma gives information about the image of a meridian of a component of $\pi(E)$

Lemma 3.2. *Let G be a compact surface, $e \in \mathscr{C}(G)$ and let $\hat{Z}$ be the spine for $\hat{X}$ constructed above. Suppose that $\psi : \hat{Z} \to M$ is an embedding into a 3-manifold M such that $\psi(T)$ bounds a solid torus U disjoint from $\psi(\hat{Z})$, and $\psi(l)$ is homotopic to the core curve of U.*

Then there exists $s \in \mathbb{Z}$ such that

(1) $\psi(m + sl)$ bounds a disk in U,
(2) $\psi \circ f_0 : G \to M$ is homotopic to $\psi \circ f_k \circ D^{ks}$ for all k, and
(3) $\psi \circ f_1$ is homotopic to $\psi \circ f_k \circ D^{(k-1)s}$ for all k

where $D : G \to G$ is a right Dehn twist about E.

Proof. Since $\psi(l)$ is homotopic to the core curve of U, it is a longitude for U. So, the meridian m_U for U will intersect $\psi(l)$ exactly once. Therefore, the pre-image $\psi^{-1}(m_U)$ of the meridian will intersect l exactly once and must be of the form $m + sl$ for some $s \in \mathbb{Z}$.

If $s = 0$ then (2) and (3) hold, since we may extend ψ to an embedding $\bar{\psi} : X \to M$ and f_0 is homotopic to f_k within X for all k.

We now define a map $h : \hat{Z} \to \hat{Z}$ which allows us to reduce to the $s = 0$ case. Let h be the identity on $\hat{Z} - A$ and let $h|_A = D_A^{-s}$ where D_A is the right Dehn twist about the core curve of A so that $h_*(m) = m + sl$. Then $\psi \circ h : \hat{Z} \to M$ is an embedding so that $\psi \circ h(T)$ bounds U and $\psi \circ h(m)$ bounds a disk in U. Therefore, for any k, $\psi \circ h \circ f_0$ is homotopic to $\psi \circ h \circ f_k$. The f_k-pre-image of A in G is a collection of k parallel annuli and the map $h \circ f_k$ is equal to pre-composing f_k with s Dehn twists in each of the k annuli. As s Dehn twists in k parallel annuli is homotopic to ks

Dehn twists in a single annulus we have that $h\circ f_k : G \to Z$ is homotopic to $f_k \circ D^{ks}$ for all k. Properties (2) and (3) follow immediately. □

3.2. **Wrapping multicurves and wrapping numbers.** In this section, we analyze how compact cores for algebraic limits immerse into geometric limits. We identify the wrapping multicurve and produce a level surface in the algebraic limit whose projection to the geometric limit is embedded off of a collar neighborhood of the wrapping multicurve. At the end of the section, we define the wrapping numbers of the wrapping multicurves.

Proposition 3.3. *Suppose that $\{\rho_n\} \subset AH(S)$, $\lim \rho_n = \rho$, and $\{\rho_n(\pi_1(S))\}$ converges geometrically to $\hat{\Gamma}$. Let $\hat{N} = \mathbb{H}^3/\hat{\Gamma}$ and let $\pi : N_\rho \to \hat{N}$ be the obvious covering map. There exists a level surface F in N_ρ, a multicurve $q = \{q_1,\dots,q_r\}$ on F, and an open collar neighborhood $Q = \mathbf{collar}(q) \subset F$, so that*

(1) π restricts to an embedding on $F - Q$.

(2) $l_\rho(q) = 0$ and if Q_i is the component of Q containing q_i, then $\pi|_{Q_i}$ is an immersion, which is not an embedding, into the boundary T_i of a cusp region V_i.

(3) If $\hat{J}$ is a (closed) regular neighborhood of $\pi(F)$ in $\hat{N}^0$, then $\hat{J}$ is homeomorphic to $F \times [-1,1] \setminus (Q \times (-\frac{1}{2},\frac{1}{2}))$ and $\partial_1 J = \partial\hat{J} - \pi(\partial N^0_\rho)$ is incompressible in $\hat{N}^0$. In particular, $\pi_1(\hat{J})$ injects into $\hat{\Gamma}$.

(4) If d is a downward-pointing parabolic in N_ρ, then $i(d,q) = 0$. Moreover, if d is not a component of q and c is a curve in S which intersects d, then the geodesic representative c^* lies above d^* in N_{ρ_n} for all large enough n.

Analogously, if d is an upward-pointing parabolic in N_ρ, then $i(d,q) = 0$. Moreover, if d is not a component of q and c is a curve in S which intersects d, then the geodesic representative c^* lies below d^* in N_{ρ_n} for all large enough n.

(5) If there is a compact core for N_ρ which embeds, under π, in $\hat{N}$, then q is empty.

We will call q the *wrapping multicurve* of the triple $(\{\rho_n\},\rho,\hat{\Gamma})$. We say that a parabolic curve d for ρ is an *unwrapped parabolic* for the triple $(\{\rho_n\},\rho,\hat{\Gamma})$ if it does not lie in the wrapping multicurve q.

Proof. Let $\widehat{\mathscr{H}}$ be an invariant collection of horoballs for the parabolic elements of $\hat{\Gamma}$ and let $\mathscr{H}$ be the subset of $\widehat{\mathscr{H}}$ consisting of horoballs based at fixed points of parabolic elements of $\rho(\pi_1(S))$. Let

$$\hat{N}^0_\rho = (\mathbf{H}^3 - \bigcup_{H\in\widehat{\mathscr{H}}} H)/\hat{\Gamma} \quad \text{and} \quad N^0_\rho = (\mathbf{H}^3 - \bigcup_{H\in\mathscr{H}} H)/\rho(\Gamma)$$

and let (M,P) be a relative compact core for N^0_ρ.

Let A be a maximal collection of disjoint, nonparallel essential annuli in (M,P) with one boundary component in P. Since one may identify M with $S\times[-1,1]$ so that $\partial S\times[-1,1]$ is identified with a collection of components of P, one may identify A with $a\times[-1,1]$ where $a = \{q_1,\dots,q_t\}$ is a disjoint collection of simple closed curves on S. Let R be the complement in S of a collar neighborhood of the multicurve a. Let $\{R_j\}$ be the components of R and let $\Gamma^j = \rho(\pi_1(R_j))$. Notice that an element of Γ^j is parabolic if and only if it is conjugate to an element of $\rho(\pi_1(\partial R_j))$. Proposition 6.4 in [13] implies that there exists a proper embedding $h : R_j \to \hat{N}^0$ such that $h_*(\pi_1(R_j))$ is conjugate to Γ^j for each j. In particular, $h(\partial R_j) \subset \partial \hat{N}^0$.

We now construct F. For each j, let F_j be a lift of $h(R_j)$ to N_ρ. For each i, let Q_i be the annulus in ∂N_ρ^0 joining two components of $\bigcup \partial F_j$ whose core curve is homotopic to q_i. Then $F = \bigcup F_j \cup \bigcup Q_i$ is a level surface for N_ρ^0.

We re-order $\{q_1, \ldots, q_r, q_{r+1}, \ldots, q_t\}$ so that if $i \le r$, then $\pi|_{Q_i}$ is not an embedding, while if $i > r$, then $\pi|_{Q_i}$ is an embedding. Let $q = \{q_1, \ldots, q_r\}$ and $Q = Q_1 \cup \cdots \cup Q_r$. Conditions (1) and (2) are satisfied by construction.

Let $\hat{J}$ be a (closed) regular neighborhood of $\pi(F)$ in $\hat{N}^0$. By construction, $\hat{J}$ is homeomorphic to $F \times [-1,1] \setminus (Q \times (-\frac{1}{2}, \frac{1}{2}))$. We first prove that

$$\partial_1 \hat{J} = \partial \hat{J} - \pi(\partial \hat{N}_\rho^1) \cong F \times \{-1, 1\}$$

is incompressible in $\hat{N}^0$. Since $\partial_1 \hat{J}$ is clearly incompressible in $\hat{J}$, we only need to check that $\partial_1 \hat{J}$ is incompressible in $\hat{N}^0 - \mathrm{int}(\hat{J})$. Each component E of $\partial_1 \hat{J}$ is homeomorphic to S. If E is not incompressible in $\hat{N}^0 - \hat{J}$, then there exists an embedded disk D in $\hat{N}^0 - \mathrm{int}(\hat{J})$ which is bounded by a homotopically non-trivial curve in $\partial_1 \hat{J}$.

By Lemma 2.12, there exists, for all large enough n, Z_n and a 2-bilipschitz embedding $\psi_n : Z_n \to N_{\rho_n}$ so that $\hat{J} \cup D \subset Z_n$ and if T is a toroidal boundary component of $\hat{J}$, then $\psi_n(T)$ bounds a Margulis tube in N_{ρ_n}. Moreover, if c is a curve in $R_j \cap T$, for some j, then $\psi_n(c)$ is homotopic to the core curve of the Margulis tube. Let J_n be the union of $\psi_n(\hat{J})$ and all the Margulis tubes bounded by toroidal components of $\psi_n(\partial \hat{J})$. Then, J_n is homeomorphic to $F \times [0,1]$ and $F_n = \psi_n(\pi(F))$ is homotopic, within J_n, to a level surface of J_n. Moreover, Lemma 2.12(4) implies that each level surface of J_n is properly homotopic to a level surface in $N_{\rho_n}^1$ for all large enough n. Hence, $\psi_n(D)$ is a disk in N_{ρ_n} bounded by a homotopically non-trivial curve in an embedded incompressible surface, which is impossible. Therefore, $\partial_1 \hat{J}$ is incompressible in $\hat{N}^0$. Since $\partial \hat{N}^0$ is incompressible in $\hat{N}$, it follows that $\pi_1(\hat{J})$ injects into $\hat{\Gamma}$. We have established property (3).

We now turn to the proof of property (4). Let d be a parabolic curve for N_ρ. If $i(d,q) \neq 0$, then $\pi(d)$ is non-peripheral in the regular neighborhood $\hat{J}$ of $\pi(F)$. Since, $\partial_1 \hat{J}$ is incompressible in $\hat{N}^0$, it follows that $\pi(d)$ is non-peripheral in $\hat{N}^0$. However, since $\pi(d)$ is associated to a parabolic element of $\hat{\Gamma}$, this is impossible. Therefore, $i(d,q) = 0$.

Now suppose that d is an unwrapped downward-pointing parabolic. It remains to show that if c is a curve on S which intersects d, then the geodesic representative of c lies above the geodesic representative d_n^* of d in N_{ρ_n} for all sufficiently large n.

We first observe that there exists an immersed annulus A in $\hat{N}^0$ joining $\pi(d)$ to an essential curve a in the cusp region $V(\pi(d))$ associated to $\pi(d)$ in $\hat{N}$ whose interior is disjoint from $\pi(F)$. We may assume that $\hat{J}$ is disjoint from $V(\pi(d))$. Since

$\pi(d)$ is homotopic into $V(d)$, there exists an essential curve a in $\partial V(d)$ which is homotopic to $\pi(d)$. Let A be an immersed annulus in $\hat{N}^0$ joining $\pi(d)$ to a. If A cannot be chosen so that its interior is disjoint from $\pi(F)$, then there exists a curve $\pi(b)$ in $\pi(F-Q)$ which is homotopic to $\pi(d)$ in $\hat{N}$, but b is not homotopic to d in F. Then there exists $\gamma \in \hat{\Gamma} - \rho(\pi_1(S))$ such that $\gamma\rho(b)\gamma^{-1} = \rho(d)$, so $\rho(b)$ is also parabolic. Let $V(b)$ and $V(d)$ be the distinct cusp regions associated to b and d in N_ρ. Since $\pi(b)$ is homotopic to $\pi(d)$, $\pi(V(b)) = \pi(V(d))$. Lemma 2.12 implies that $\rho_n(b)$ is homotopic to $\rho_n(d)$ in N_{ρ_n} for all large enough n, which is a contradiction. Therefore, we may assume that the interior of A is disjoint from $\pi(F)$ as claimed.

We next observe that $F_n = \psi_n(\pi(F))$ lies above d_n^* for all large enough n. The annulus A lifts to an annulus in N_ρ which lies below F. We may assume that, for all large enough n, $A \subset Z_n$ and $\psi_n(a)$ is an essential curve in the boundary of the Margulis tube associated to d_n^* in N_{ρ_n}. Let R be the component of $F-Q$ containing d. Since $\ell_\rho(\partial R) = 0$, a result of Otal [36, Theorem A] implies that, for all large enough n, the geodesic representative of each component of ∂R is unknotted in N_{ρ_n}. Lemma 2.9 in [12] then implies that $\psi_n(\pi(R))$ is a level subsurface of $N^1_{\rho_n}$ for all large enough n. Since $\psi_n(A)$ lies below $\psi_n(\pi(R))$, d_n^* lies below the embedded subsurface $\psi_n(\pi(R))$. Lemma 2.7 in [12] then implies that d_n^* also lies below F_n.

If c is a curve on S which intersects d essentially, then c has a representative c_n on F_n of length at most $L(c)$, for all n, so there exists a homotopy from c_n to either c_n^* or to a Margulis region in N_{ρ_n} associated to c which has tracks of length at most $D(c)$, where $D(c)$ depends only on $L(c)$ (see [13, Lemma 2.6]). If $d(\partial \mathbb{T}^n_\varepsilon, d_n^*) > D(c)$, then this homotopy will miss d_n^*, which implies that c_n^* lies above d_n^*. However, this will be the case if $\ell_{\rho_n}(d)$ is sufficiently close to 0, which occurs for all large enough n.

The proof of property (4) for unwrapped upward-pointing parabolics is analogous.

If q is non-empty and there is a compact core for N which embeds in N_ρ, then $\pi|_F$ is homotopic to an embedding. However, since $\partial_1 \hat{J}$ has incompressible boundary, this implies that $\pi|_F$ is homotopic to an embedding within $\hat{J}$, which is clearly impossible. This establishes (5) and completes the proof. □

Let q_i be a curve of q. We will now define the wrapping number of q_i with respect to $(\{\rho_n\}, \rho, \Gamma)$. Consider the manifolds X and $\hat{X}$ defined in section 3.1 with $G = F$ and $e = q_i$. From (3) we get an inclusion $\iota : \hat{J} \to \hat{X}$. Furthermore, $\iota \circ \pi : F \to \hat{X}$ is homotopic, as a map into X, to $id \times \{0\}$. Lemma 3.1 implies that there is a unique $k \in \mathbb{Z}$ such that $\iota \circ \pi$ is homotopic to f_k as maps into $\hat{X}$. We then define the *wrapping numbers* $w^+(q_i) = k$ and $w^-(q_i) = k-1$. Of course, it is clear that w^+ determines w^-, but as we will see, it is convenient to keep track of both numbers. Notice that the parabolic corresponding to a curve q_i is downward pointing (in N_ρ) if and only if $w^+(q_i) > 0$.

If $q = \{q_1, \ldots, q_r\}$, then we get r-tuples

$$w^+(q) = (w^+(q_1), \ldots, w^+(q_r)) \quad \text{and} \quad w^-(q) = (w^-(q_1), \ldots, w^-(q_r)).$$

Remarks: (1) Our wrapping numbers are closely related to the wrapping coefficients discussed in Brock-Canary-Minsky [13, Section 3.6] and the wrapping numbers defined by Evans-Holt [18].

(2) Proposition 3.3 may be viewed as a special case (and amplification) of the analysis carried out in section 4 of Anderson-Canary-McCullough [3]. In the language of that paper, the subsurfaces $\{F_j\}$ are the relative compact carriers of the precisely embedded system $\{\rho(\pi_1(R_j))\}$ of generalized web subgroups.

4. Asymptotic behavior of end invariants in convergent sequences

In this section, we prove that if a sequence of Kleinian surface groups converges, then some subsequence of the end invariants bounds projections. Along the way we will see that the sequence of end invariants also predicts the parabolics in the algebraic limit and whether they are upward-pointing or downward-pointing. In combination with results from [12] we see that the asymptotic behavior of the end invariants predicts all the lamination and curve components of the end invariants of the algebraic limit. Predicting the conformal structures which arise is significantly more mysterious. We also see that the asymptotic behavior of the end invariants predicts the wrapping multicurve and the associated wrapping numbers in any geometric limit.

Theorem 4.1. *Suppose that $\{\rho_n\}$ is a convergent sequence in $AH(S)$ with end invariants $\{\nu_n^\pm\}$ and that $\lim \rho_n = \rho$. Then there exists a subsequence $\{\rho_j\}$ such that the sequence $\{\nu_j^\pm\}$ bounds projections.*

Furthermore, if $\{\rho_j\}$ is a subsequence such that $\{\nu_j^\pm\}$ bounds projections, then

1. *$\ell_\rho(d) = 0$ if and only if d is a combinatorial parabolic for the sequence $\{\nu_j^\pm\}$,*
2. *A parabolic curve d is upward-pointing in N_ρ if and only if*
$$|m(\nu_j^+,d,\mu)| - |m(\nu_j^-,d,\mu)| \to +\infty.$$
3. *A lamination $\lambda \in \mathcal{EL}(Y)$ is an ending lamination for an upward-pointing (respectively downward-pointing) geometrically infinite end for N_ρ if and only if $\{\pi_Y(\nu_j^+)\}$ (respectively $\{\pi_Y(\nu_j^-)\}$) converges to $\lambda \in \mathcal{C}(Y) \cup \mathcal{EL}(Y)$.*
4. *If $\{\rho_j(\pi_1(S))\}$ converges geometrically to $\hat{\Gamma}$, then the wrapping multicurve for $(\{\rho_j\},\rho,\Gamma)$ is the collection of combinatorial wrapping parabolics given by $\{\nu_j^\pm\}$ and if d is a wrapping parabolic, then the combinatorial wrapping number $w(d)$ agree with the actual wrapping number $w^+(d)$.*

Remark: In general, it is necessary to pass to a subsequence since the phenomenon of self-bumping (see McMullen [30] or Bromberg-Holt [14]) assures that you can have a convergent sequence with one subsequence where the wrapping multicurve is empty and another subsequence where the wrapping multicurve is non-empty.

Proof. We first prove that any geodesic joining ν_n^+ to ν_n^- always intersects some bounded set.

Lemma 4.2. *Suppose that $\{\rho_n\}$ is a sequence in $AH(S)$ converging to ρ. Let $\nu_n^\pm$ be the end invariants of ρ_n. There is a bounded set $\mathscr{B} \subset \mathscr{C}(S)$ such that every geodesic joining $\pi_S(\nu_n^+)$ to $\pi_S(\nu_n^-)$ intersects $\mathscr{B}$.*

Proof. If a is any curve in $\mathscr{C}(S)$, then there is a uniform upper bound L_a on the length $l_{\rho_n}(a)$ for all n. It follows from Theorem 2.6 that a lies within $D(L_a) = D_0(\max\{L_a, L_0\})$ of any geodesic joining $\pi_S(\nu_n^+)$ to $\pi_S(\nu_n^-)$. One may thus choose $\mathscr{B}$ to be a neighborhood of a in $\mathscr{C}(S)$ of radius $D(L_a)$. □

We next show that $\ell_\rho(d)$ is non-zero if and only if $\{m(\nu_n^+, \mu, d)\}$ and $\{m(\nu_n^-, \mu, d)\}$ are both eventually bounded for any marking μ. This is a fairly immediate consequence of work of Minsky, namely Theorem 2.5 and the Short Curve Theorem of [32].

Lemma 4.3. *Suppose that $\{\rho_n\}$ is a sequence in $AH(S)$ converging to ρ. Let $\nu_n^\pm$ be the end invariants of ρ_n and let μ be a marking on S. Then, $l_\rho(d) > 0$ if and only if $\{m(\nu_n^+, d, \mu)\}$ and $\{m(\nu_n^-, d, \mu)\}$ are both eventually bounded.*

Proof. First suppose that $\{m(\nu_n^\beta, d, \mu)\}$ is not eventually bounded, then either $\{\frac{1}{l_{\nu_n^\beta}(d)}\}$ is not eventually bounded or $\{\sup_{d\subset\partial Y} d_Y(\nu_n^\beta, \mu)\}$ is not eventually bounded. If $\{\frac{1}{l_{\nu_n^\beta}(d)}\}$ is not eventually bounded, then there exists a subsequence $\{\rho_j\}$ of $\{\rho_n\}$ so that $\{\ell_{\rho_j}(d)\}$ converges to 0, which implies that $\ell_\rho(d) = 0$. If $\{\sup_{d\subset\partial Y} d_Y(\nu_n^\beta, \mu)\}$ is not eventually bounded, then either

(1) there exists a subsequence for which d is always a component of ν_j^β, or

(2) there exists a sequence of subsurfaces Y_j such that $d \subset \partial Y_j$ and $d_{Y_j}(\nu_j^\beta, \mu) \to \infty$.

In case (1), $\ell_{\rho_j}(d) = 0$ for all j, so $\ell_\rho(d) = 0$. In case (2), since $\ell_{\rho_j}(\mu)$ is eventually bounded and $\pi_{Y_j}(\mu_j^\beta) \in \overline{\pi_{Y_j}(\mathscr{C}(\rho, L_B))}$ where L_B is the Bers constant for S (see Brock-Bromberg-Canary-Minsky [12, Section 2]), we see that $\mathrm{diam}(\pi_{Y_j}(\mathscr{C}(\rho, L)) \to \infty$ for some L. Theorem 2.5 then implies that $\lim \ell_{\rho_j}(d) = 0$, so again $\ell_\rho(d) = 0$. Therefore, in all cases, if $\{m(\nu_n^\beta, d, \mu)\}$ is not eventually bounded, then $\ell_\rho(d) = 0$. It follows that if $l_\rho(d) > 0$ then $\{m(\nu_n^+, d, \mu)\}$ and $\{m(\nu_n^-, d, \mu)\}$ are both eventually bounded.

If $\ell_\rho(d) = 0$, then Minsky's Short Curve Theorem [32] implies that at least one of $\{\frac{1}{\ell_{\nu_n^+}(d)}\}$, $\{\frac{1}{\ell_{\nu_n^-}(d)}\}$, and $\{\sup_{d\subset\partial Y} d_Y(\nu_n^+, \nu_n^-)\}$ is not eventually bounded. (For a similar restatement of the Short Curve Theorem in the quasifuchsian case see Brock-Bromberg-Canary-Minsky [11, Thm. 2.2].) It follows that $\{m(\nu_n^\beta, d, \mu)\}$ is not eventually bounded for some $\beta \in \{\pm\}$. □

We now pass to a subsequence $\{\rho_j\}$ of $\{\rho_n\}$ so that $\{\rho_j(\pi_1(M))\}$ converges geometrically to $\hat{\Gamma}$. Let $\hat{N} = \mathbf{H}^3/\Gamma$. Let F, $q = \{q_1, \ldots, q_s\}$ and Q, be the level surface, wrapping multicurve and collar neighborhood of q provided by Proposition 3.3. Let

$$f : S \to F$$

be a homeomorphism such that $f_* = \rho \in AH(S)$ and let $\hat{J}$ be a closed regular neighborhood of $\pi(F)$ in $\hat{N}^0$.

The following lemma characterizes the asymptotic behavior of the end invariants relative to an unwrapped parabolic.

Lemma 4.4. *Suppose that $\{\rho_j\}$ is a sequence in $AH(S)$ converging to ρ such that $\{\rho_j(\pi_1(S))\}$ converges geometrically to $\hat{\Gamma}$ and that d is an unwrapped parabolic for the triple $(\{\rho_j\},\rho,\hat{\Gamma})$. Then,*

1. *(1) if d is a downward-pointing cusp in N_ρ, then $\{m(\nu_j^+,d,\mu)\}$ is eventually bounded, and*
2. *(2) if d is an upward-pointing cusp in N_ρ, then $\{m(\nu_j^-,d,\mu)\}$ is eventually bounded.*

Proof. Let d be a downward-pointing cusp in N_ρ and let L_μ be an upper bound for the length of μ in N_{ρ_j} for all j. Let a be a curve in μ which crosses d. Proposition 3.3 guarantees that a^* lies above d^* in N_{ρ_j} for all sufficiently large j. Theorem 2.8 then implies that, for all sufficiently large j, $d_Y(a,\nu_j^+) < D(L_\mu)$ if $Y \subset S$ is any subsurface with d in its boundary. Therefore, $\{\sup_{d\subset\partial Y} d_Y(\nu_j^+,\mu\})$ is eventually bounded.

It remains to check that there is an eventual lower bound on $l_{\nu_j^+}(d)$. (For a similar argument in the quasifuchsian case, see [11, Lemma 2.5].) Since $l_{\rho_j}(a) < L_\mu$ for all j, there exists $\varepsilon > 0$ so that the geodesic representative a_j^* of a in N_{ρ_j} misses $\mathbb{T}_\varepsilon(d)$ for all j. The convex core $\partial C(N_{\rho_j})$ of N_{ρ_j} is the smallest convex submanifold of N_{ρ_j} containing all the closed geodesics. Epstein, Marden and Markovic [17, Theorem 3.1] proved that there is a 2-Lipschitz map $f_j : \partial_c N_{\rho_j} \to \partial C(N_{\rho_j})$ so that f_j extends to a strong deformation retraction of $\partial_c N_{\rho_j} \cup N_{\rho_j}$ onto $C(N_{\rho_j})$. In particular, if R_j is a downward-pointing component of $\partial_c N_{\rho_j}$, then no closed geodesic in N_{ρ_j} lies below $f(R_j)$. If $l_{\nu_j^+}(d) = l_j < \varepsilon/2$, then there is a representative d_j of d in the image $f_j(R_j)$ of a downward-pointing component R_j of $\partial_c N_{\rho_j}$ which has length at most $2l_j < \varepsilon$, so is contained in $\mathbb{T}_\varepsilon(d_j)$. Therefore, a_j^* cannot intersect d_j, so is disjoint from $f(R_j)$. It follows that $f(R_j)$ lies below a_j^*, which implies that d_j lies below a_j^*. Since d_j is homotopic to d_j^* within $\mathbb{T}_\varepsilon(d_j)$ and a_j^* is disjoint from $\mathbb{T}_\varepsilon(d_j)$, we see that d_j^* lies below a_j^*. However, this contradicts Proposition 3.3, so $l_{\nu_j^+}(d) \geq \varepsilon/2$ for all sufficiently large j which completes the proof for downward-pointing cusps.

The proof in the case that d is an upward-pointing cusp is similar. □

The situation is more complicated for wrapped parabolics.

We will abuse notation by letting q also denote the multicurve $f^{-1}(q) \subset S$ and by letting Q denote the subsurface $f^{-1}(Q)$ of S. Let $X = S \times [-1,1]$ and $\hat{X} = X - V$ where $V = Q \times (-\frac{1}{2},\frac{1}{2}) \subset X$ is a union of open solid tori in the homotopy class of q. Set

$$\hat{Z} = (S - Q) \times \{0\} \cup \partial V \subset \hat{X}.$$

If q is a single curve, then we are in the situation of section 3.1 with $G = S$ and $e = q$. We encourage the reader to focus on this situation when first reading the section.

In general, if $q = \{q_1, \ldots, q_r\}$, we divide S into a collection of overlapping subsurfaces $\{G_1, \ldots, G_r\}$ defined as follows: G_i is the connected component of $S - (Q - Q_i)$ that contains Q_i. One may then divide X up into overlapping submanifolds $\{X_1, \ldots, X_r\}$ where $X_i = G_i \times [-1, 1]$. Similarly, one may divide $\hat{X}$ up into submanifolds $\{\hat{X}_1, \ldots, \hat{X}_r\}$ with $\hat{X}_i = X_i - V_i$ where $V_i = Q_i \times (-\frac{1}{2}, \frac{1}{2}) \subset X_i$. Let T_i be the toroidal boundary component of $\hat{X}_i$.

Proposition 3.3 implies that we may identify $\hat{J}$ with $\hat{X}$. If $k = (k_1, \ldots, k_r)$ we may define a map $f_k : S \to \hat{J}$ which agrees on each G_i with the map $f_{k_i} : G_i \to \hat{X}_i$ defined in section 3.1. Each of the f_k determines a representation $(f_k)_*$ in $AH(S)$. Since f is homotopic to $f_{w^+(q)}$, we see that $(f_{w^+(q)})_* = \rho$.

Given a component Q_i of Q we denote by $D_i : S \to S$ the right Dehn twist about Q_i. For an r-tuple $k = (k_1, \ldots, k_r)$, we set $D_q^k = D_1^{k_1} \circ \ldots \circ D_r^{k_r}$.

Lemma 4.5. *For all large enough j, there exists a r-tuple $s_j = (s_{1,j}, \ldots, s_{r,j})$ such that $\rho_j^+ = \rho_j \circ (D_q^{w^+(q)s_j})_*$ and $\rho_j^- = \rho_j \circ (D_q^{w^-(q)s_j})_*$ have the following properties:*

1. *the sequences $\{\rho_j^+\}$ and $\{\rho_j^-\}$ converge in $AH(S)$ to ρ^+ and ρ^-.*
2. *If q_i is a component of q, then q_i is an upward pointing parabolic in N_{ρ^-} and a downward pointing pararabolic in N_{ρ^+} and is unwrapped in the triples $(\{\rho_j^+\}, \rho^+, \hat{\Gamma})$ and $(\{\rho_j^-\}, \rho^-, \hat{\Gamma})$.*
3. *For each i, $\lim |s_{i,j}| = +\infty$.*

Proof. For all large enough j, there exists a 2-bilipschitz embedding $\psi_j : \hat{J} \to N_{\rho_j}$ such that each component of $\psi_j \circ \phi(\partial V)$ bounds a Margulis tube in N_{ρ_j} and $\psi_j(\hat{J})$ is disjoint from the interior of these tubes. Moreover, $(\psi_j \circ \hat{f})_*$ is conjugate to ρ_j. In particular, if l is the longitude of any component T of ∂V, then $\psi_j(l)$ is a longitude of the Margulis tube bounded by $\psi_j(T)$.

Given $j \in \mathbb{N}$, Lemma 3.2 applied to $G = G_i$ and $e = q_i$ implies that for all i, there exists $s_{i,j}$ so that if m_i and l_i are the meridian and longitude of T_i, then $\psi_j(m_i + s_{i,j} l_i)$ bounds a meridian of $\psi_j(T_i)$. We set $f^- = f_{(0,\ldots,0)}$ and $f^+ = f_{(1,\ldots,1)}$ and let $\rho^+ = f_*^+$ and $\rho^- = f_*^-$. Lemma 3.2 implies that $\psi_j \circ f^+$ is homotopic to $f \circ D_q^{w^+(q)s_j}$ and that $\psi_j \circ f^+$ is homotopic to $f \circ D_q^{w^-(q)s_j}$. It follows that $\{\rho_j^+\}$ converges to ρ^+ and that $\{\rho_j^-\}$ converges to ρ^-. This establishes property (1) and property (2) is true by construction.

It remains to establish property (3). Notice that since $\lim \ell_{\rho_j}(q_i) = 0$, the diameter of the Margulis tube bounded by $\psi_j(T_i)$ is diverging to $+\infty$. It follows that the length of the meridian of $\psi_j(T_i)$ diverges to $+\infty$. Since ψ_j is 2-bilipschitz, there is a uniform upper bound on the lengths of $\psi_j(l_i)$ and $\psi_j(m_i)$. Since the meridian of $\psi_j(T_i)$ is homotopic to $\psi_j(m_i + s_{i,j} l_i)$, we must have $\lim |s_{i,j}| = +\infty$. □

We can now easily assemble the proof of Theorem 4.1. We first show that if $\{\rho_n\}$ converges, then there is a subsequence $\{\rho_j\}$ so that $\{\nu_j^\pm\}$ bounds projections. We choose a subsequence so that $\{\rho_j(\pi_1(S))\}$ converges geometrically. Lemma 4.2 implies that $\{\nu_j^\pm\}$ satisfies condition (a) of the definition of bounding projections. Lemma 4.3 implies that all the curves d which are not parabolic in the algebraic limit satisfy condition (b)(i). Lemma 4.4 implies that if d is an unwrapped parabolic, then it satisfies condition (b)(i), while Lemma 4.5 combined with Lemma 4.4 implies that any wrapped parabolic curve d satisfies condition (b)(ii). Therefore, $\{\nu_j^\pm\}$ bounds projections as claimed.

We now suppose that $\{\rho_j\}$ is a subsequence so that $\{\rho_j(\pi_1(S))\}$ converges geometrically. Property (1) follows from Lemma 4.3. Property (2) follows from Lemma 4.4 if d is an unwrapped parabolic. Property (2) for wrapped parabolics follows from Lemma 4.4 and the facts, observed in section 3.1, that $w^-(q) = w^+(q) - 1$ and that d is upward pointing if and only if $w^+(q)$ is positive. Property (3) comes from Theorem 2.4 ([12, Theorem 1.1]). Property (4) follows from Lemma 4.5.

In general, if $\{\rho_j\}$ is a subsequence of $\{\rho_n\}$ so that $\{\nu_j^\pm\}$ bounds projections. Then every subsequence of $\{\rho_j\}$ has a subsequence $\{\rho_k\}$ so that $\{\rho_k(\pi_1(S))\}$ converges geometrically. Therefore, every subsequence of $\{\rho_j\}$ has a subsequence for which properties (1)–(4) hold. It is then easily checked that properties (1)–(4) hold for the original sequence $\{\rho_j\}$. □

5. MULTICURVES FROM END INVARIANTS

In this section, we prove that if the sequence of end invariants bounds projections, then we can find a sequence of pairs of bounded length multicurves which bounds projections.

Proposition 5.1. *Suppose that $\{\rho_n\}$ is a sequence in $AH(S)$ with end invariants $\{\nu_n^\pm\}$. If $\{\nu_n^\pm\}$ bounds projections, then there exists a subsequence $\{\rho_j\}$ and a sequence of pairs of multicurves $\{c_j^\pm\}$ such that $\{\ell_{\rho_j}(c_j^+ \cup c_j^-)\}$ is bounded and $\{c_j^\pm\}$ bounds projections.*

The moral here is quite simple, although unpleasant technical difficulties arise in the actual proof. If $\{\rho_n\}$ is a sequence of quasifuchsian groups, one might hope to be able to choose c_n^+ and c_n^- to be minimal length pants decompositions of the top and bottom conformal boundaries of N_n. There are three technical issues that cause this simple algorithm to fail:

- The c_n^+ and c_n^- cannot have curves in common.
- A downward (upward) pointing unwrapped combinatorial parabolic cannot be in c_n^+ (c_n^-).
- A wrapped combinatorial parabolic cannot be in either c_n^+ or c_n^-.

It is easy to construct examples where the minimal length pants decompositions fail to satisfy any of these technical constraints. To deal with these issues, we will choose c_n^+ to be a minimal length pants decomposition of ν_n^+ which intersects any downward-pointing combinatorial parabolic, any combinatorial wrapped parabolic and any "sufficiently short" curve on ν_n^-. We then choose c_n^- to be a minimal length pants decomposition of ν_n^- which intersects any curve in c_n^+, any downward-pointing combinatorial parabolic, and any combinatorial wrapped parabolic.

In general, one might hope to choose c_n^+ to consist of a minimal length pants decomposition of each geometrically finite subsurface on the "top," a curve for each upward-pointing parabolic and a pants decomposition of each subsurface supporting an upward-pointing geometrically infinite end which is "close enough" to the ending lamination. We will again need to be more careful in the actual proof.

Proof. We first pass to a subsequence, still called $\{\rho_n\}$, so that if d is a curve and $\beta \in \{\pm\}$, then either $m(\nu_n^\beta, d, \mu) \to \infty$ or $\{m(\nu_n^\beta, d, \mu)\}$ is eventually bounded. Let b^β be the collection of curves such that $m(\nu_n^\beta, d, \mu) \to \infty$ if and only if d is in b^β. If d lies in b^+ or b^-, then d is a combinatorial parabolic, while if d lies in both b^+ and b^-, then d is a combinatorial wrapped parabolic.

The following lemma implies that b^+ and b^- are multicurves.

Lemma 5.2. *Suppose that $\{\rho_n\}$ is a sequence in $AH(S)$ with end invariants $\{\nu_n^\pm\}$ and $\{\nu_n^\pm\}$ bounds projections. If d is either an upward-pointing or wrapped combinatorial parabolic and c intersects d, then $\{m(\nu_n^+, c, \mu)\}$ is eventually bounded.*

Similarly, if d is either a downward-pointing or wrapped combinatorial parabolic and c intersects d, then $\{m(\nu_n^-, c, \mu)\}$ is eventually bounded.

Proof. We give the proof in the case that d is either an upward-pointing combinatorial parabolic or a combinatorial wrapped parabolic, in which case $m(\nu_n^+, d, \mu) \to \infty$. The proof of the other case is analogous.

First suppose that $\ell_{\nu_n^+}(d) \to 0$, so $l_{\nu_n^+}(c) \to \infty$ and d is a curve in the base of the (generalized) marking $\mu(\nu_n^+)$ (defined in section 2.2) associated to ν_n^+ for all large enough n. In particular, if $c \in \partial Z$, then

$$d_Z(\mu, \nu_n^+) \leq d_Z(\mu, d) + d_Z(d, \mu(\nu_n^+)) \leq 2i(\mu, d) + 6.$$

(The second inequality follows from Lemma 2.1 and the fact that any two curves in $\mu(\nu_n^+)$ intersect at most twice.) Therefore, if $\ell_{\nu_n^+}(d) \to 0$, then $\{m(\nu_n^+, c, \mu)\}$ is eventually bounded.

Notice that, by reversing the roles of c and d in the previous sentence, we see that if $m(\nu_n^+, d, \mu) \to \infty$, then $\{\ell_{\nu_n^+}(c)\}$ is bounded away from zero.

So, we may suppose that both $\{\ell_{\nu_n^+}(d)\}$ and $\{\ell_{\nu_n^+}(c)\}$ are bounded away from zero, and that $\sup_{d \subset \partial Y} d_Y(\nu_n^+, \mu) \to \infty$. Therefore, there exists a sequence of subsurfaces Y_n with $d \subset \partial Y_n$, so that $d_{Y_n}(\nu_n^+, \mu) \to \infty$. It follows that $d_{Y_n}(\nu_n^+, c) \to \infty$. Lemma 2.2 then implies that if Z is a subsurface with $c \in \partial Z$, then

$$d_Z(\partial Y_n, \nu_n^+) \leq 4$$

for all large enough n. So,

$$d_Z(\nu_n^+, \mu) \leq d_Z(\partial Y_n, \nu_n^+) + d_Z(\partial Y_n, \mu) \leq 4 + d_Z(d, \mu) + 1$$

for all large enough n. Since $d_Z(d, \mu)$ is bounded above by a function of $i(d, \mu)$, $\{\sup_{c \subset \partial Z} d_Z(\nu_n^+, \mu)\}$ is eventually bounded. Therefore, again $\{m(\nu_n^+, c, \mu)\}$ is eventually bounded. □

We next claim that a curve cannot be "short" on both the top and the bottom.

Lemma 5.3. *If $\{\rho_n\}$ is a sequence in $AH(S)$ with end invariants $\{\nu_n^\pm\}$ and $\{\nu_n^\pm\}$ bounds projections, then there exists $\delta_1 > 0$, so that if d is any curve on S, then*

$$\max\{\ell_{\nu_n^+}(d), \ell_{\nu_n^-}(d)\} > \delta_1.$$

Proof. If not, we may pass to a subsequence so that there exist curves a_n so that $\ell_{\nu_n^+}(a_n) + \ell_{\nu_n^-}(a_n) \to 0$. Then a_n is a curve of $\mu(\nu_n^\pm)$ hence $d_Y(a_n, \nu_n^\pm) \leq 5$ for any subsurface Y that intersects a_n essentially. If $\{a_n\}$ admits a constant subsequence a, then $m^{na}(\nu_n^+, a, \mu) \to \infty$ and $m^{na}(\nu_n^-, a, \mu) \to \infty$ which is not allowed by condition (b) of the definition of bounding projections. If not, by Lemma 2.3, there is a subsurface Y such that, after taking a subsequence, $d_Y(\mu, a_n) \to \infty$. Then we have $d_Y(\mu, \nu_n^\pm) \to \infty$ and $d_Y(\nu_n^+, \nu_n^-) \leq 5$ which contradicts both conditions (b)(i) and (b)(ii). Therefore, no such subsequence can exist and we obtain the desired inequality. □

We recall that the Collar Lemma ([15, Theorem 4.4.6]) implies that any two closed geodesics of length at most $2\sinh^{-1}(1)$ on any hyperbolic surface cannot intersect. Let e_n^β denote the multicurve on S consisting of curves d such that

$$\ell_{\nu_n^\beta}(d) < \min\{2\sinh^{-1}(1), \delta_1\}.$$

We now describe the construction of $c_n^\pm$ in the case that $\{\rho_n\}$ is a sequence of quasifuchsian representations, so $\nu_n^\pm \subset \mathscr{T}(S)$ for all n. Among the pants decompositions of S which cross every curve in $b^- \cup e_n^-$, choose one, c_n^+, with minimal length in ν_n^+ . Then among the pants decompositions of S which cross every curve in $b^+ \cup c_n^+$ choose one, c_n^-, with minimal length in ν_n^-. We observe that the resulting sequences have bounded length.

Lemma 5.4. *The sequences $\{l_{\rho_n}(c_n^+)\}$ and $\{l_{\rho_n}(c_n^-)\}$ are both bounded.*

Proof. Notice that since $\{m^{na}(\nu_n^{-\beta}, d, \mu)\}$ is bounded for all $d \in b^\beta$ and b^β has finitely many components, there exists $\delta_2 > 0$ such that if $d \in b^\beta$, then

$$\ell_{\nu_n^{-\beta}}(d) > \delta_2.$$

Lemma 5.3 implies that if d is a component of e_n^β, then $\ell_{\nu_n^{-\beta}}(d) \geq \delta_1$.

Therefore, there is a lower bound, $\min\{\delta_2, \delta_1\}$, on the length, in ν_n^+, of every curve in $b^- \cup e_n^-$. Since $b^- \cup e_n^-$ contains a bounded number of curves, it is an easy exercise to check that there is an upper bound on the length of a minimal length pants decomposition of ν_n^+ intersecting $b^- \cup e_n^-$, hence an upper bound on the length, in ν_n^+, of c_n^+.

Since c_n^+ crosses every curve in e_n^-, every curve in c_n^+ has length, in ν_n^-, at least $\min\{2\sinh^{-1}(1), \delta_1\}$. Therefore, there is a lower bound, $\min\{\delta_2, \delta_1, 2\sinh^{-1}(1)\}$, on the length, in ν_n^-, of every curve in $c_n^+ \cup b^+$. It again follows that there is an upper bound on the length of c_n^-.

Bers [6, Theorem 3] proved that if d is any curve on S, then

$$\ell_{\rho_n}(d) \leq 2\ell_{\nu_n^\beta}(d)$$

for either $\beta = +$ or $\beta = -$. It follows that both $\{l_{\rho_n}(c_n^+)\}$ and $\{l_{\rho_n}(c_n^-)\}$ are bounded. □

Since c_n^β and the base of the marking $\mu(\nu_n^\beta)$ both have uniformly bounded length in ν_n^β, there is a uniform upper bound on the intersection number between c_n^β and any base curve of the marking $\mu(\nu_n^\beta)$. Therefore, Lemma 2.1 implies that there exists K so that if $Y \subseteq S$ is not a component of $\mathbf{collar}(c_n^\beta)$ or $\mathbf{collar}(\mathrm{base}(\nu_n^\beta))$, then

$$d_Y(c_n^\beta, \nu_n^\beta) \leq K. \tag{5.1}$$

If Y is a component of $\mathbf{collar}(\mathrm{base}(\nu_n^\beta))$ and c_n^β crosses Y, then, since c_n^β has bounded length, there is a lower bound on the length of the core curve of Y and hence an upper bound on the length of the transversal to Y in the marking $\mu(\nu_n^\beta)$. Again, this implies an upper bound on the intersection number between the transversal and c_n^β, so inequality (5.1) still holds.

Finally, we pass to a subsequence so that, for each β, if d is any curve then d either lies in c_n^β for all n or for only finitely many n. Since c_n^β is a pants decomposition and c_n^- crosses every curve in c_n^+, then for any curve d there exists $\beta(d) \in \{\pm\}$ and $N(d) \in \mathbb{Z}$ such such that c_n^β crosses d for all $n \geq N(d)$.

The next lemma shows that the properties we have established suffice to show that $\{c_n^\pm\}$ bounds projections. We give the statement and the proof in the general case (i.e. ρ_n is not assumed to be quasifuchsian).

Lemma 5.5. *Let $\{\nu_n^\pm\}$ be a sequence of pairs of end invariants which bounds projections and let $\{c_n^\pm\}$ be a sequence of pairs of multicurves on S such that*

(1) there exists $K' > 0$ such that $d_S(c_n^\beta, \nu_n^\beta) \leq K'$,

(2) there exists $K > 0$ such that if $d \in \mathscr{C}(S)$, then there exists $M(d) \in \mathbb{N}$ such that if $Y \subset S$ with $d \subset \partial Y$, c_n^β crosses d, then

$$d_Y(\nu_n^\beta, c_n^\beta) \leq K, \tag{5.2}$$

for any $\beta \in \{\pm\}$ and any $n \geq M(d)$,

(3) if d is a wrapped combinatorial parabolic, then c_n^β intersects d for any $\beta \in \{\pm\}$,

(4) if d is an unwrapped downward (respectivley upward) pointing combinatorial parabolic, then c_n^+ (resp. c_n^-) intersects d, and

(5) if d is not a combinatorial parabolic, then there exists $\beta(d) \in \{\pm\}$ and $N(d) \in \mathbb{N}$ such that $c_n^{\beta(d)}$ crosses d for all $n \geq N(d)$.

Then $\{c_n^\pm\}$ bounds projections.

Proof. Since $\{\nu_n^\pm\}$ bounds projection, there exists a bounded set $\mathscr{B}$ so that any geodesic joining $\pi_S(\nu_n^+)$ to $\pi_S(\nu_n^-)$ intersects $\mathscr{B}$. By property (1), $d_S(c_n^\beta, \nu_n^\beta)$ is uniformly bounded, so the hyperbolicity of the curve complex implies that any geodesic joining c_n^+ to c_n^- lies a bounded Hausdorff distance from a geodesic joining

$\pi_S(\nu_n^+)$ to $\pi_S(\nu_n^-)$, and hence lies a bounded distance from $\mathscr{B}$. Therefore, any geodesic joining c_n^+ to c_n^- intersects some bounded set $\mathscr{B}'$, so $\{c_n^{\pm}\}$ satisfies condition (a) in the definition of bounding projections.

If d is a combinatorial wrapped parabolic, then d crosses both c_n^+ and c_n^- (by property (3)), so inequality (5.2) implies that d is a combinatorial wrapped parabolic for $\{c_n^{\pm}\}$.

If d is an unwrapped combinatorial parabolic, then there exists $\beta = \beta(d)$ so that $d \in b^{-\beta}$, so $\{m(\nu_n^{\beta(d)}, d, \mu)\}$ is eventually bounded and d crosses $c_n^{\beta(d)}$ for all n (by property (4).) Inequality (5.2) implies that $\{m(c_n^{\beta(d)}, d, \mu)\}$ is eventually bounded, so d satisfies condition (b)(i).

If d is not a combinatorial parabolic, then there exists $\beta = \beta(d)$ and $N(d)$ such that d crosses c_n^{β} for all $n \geq N(d)$ (by property (5)). Then, since $\{m(\nu_n^{\beta}, d, \mu)\}$ is eventually bounded, inequality (5.2) implies that $\{m(c_n^{\beta}, d, \mu)\}$ is eventually bounded, so d satisfies condition (b)(i). This completes the proof that condition (b) holds for every curve. □

In the quasifuchsian case, Lemma 5.4, inequality (5.1) and Lemma 5.5 imply that $\{c_n^{\pm}\}$ bounds projections, so we have completed the proof of Proposition 5.1 in the quasifuchsian case.

We next suppose that there exists a subsequence $\{\rho_n\}$ such that for all n, neither ν_n^+ or ν_n^- is a lamination supported on all of S. We list all the simple closed curves on S by fixing a bijection $\alpha : \mathscr{C}(S) \to \mathbb{N}$.

When choosing the c_n^+ on a subsurface W that supports a conformal structure in ν_n^-, we will use a procedure similar to the one used in the quasifuchsian case. If W supports a lamination λ in ν_n^+, we choose a pants decomposition that has bounded length and is "close" to λ, where close is taken to mean that the curves in the pants decomposition lie above any short curve in ν_n^- and any of the first n curves in our list that overlap W. This will allow us to establish Properties (1)–(5) in Lemma 5.5. We now make this precise.

Let c_n^+ contain every simple closed curve component of ν_n^+. If W is a subsurface which supports a conformal structure in ν_n^+, let $c_n^+|_W$ be a minimal length pants decomposition of W which intersects every component of $b^- \cup e_n^-$ which overlaps W. If the subsurface W is the support of a lamination in ν_n^+, let $c_n^+|_W$ be a pants decomposition of W of length at most L_1 in N_{ρ_n}, so that each curve in $c_n^+|_W$ lies above every curve in $\alpha^{-1}([0,n]) \cup e_n^-$ which overlaps W (see Lemma 2.9 for the existence of such a pants decomposition).

Similarly, we define c_n^- so that it contains every closed curve component of ν_n^-. If W is a subsurface which supports a conformal structure in ν_n^-, let $c_n^-|_W$ be a minimal length pants decomposition of W which intersects every component of $b^+ \cup c_n^+$ which overlaps W. If the subsurface W is the support of a lamination in ν_n^-, let $c_n^-|_W$ be a pants decomposition of W of length at most L_1 so that each curve in $c_n^-|_W$ lies below every curve in $\alpha^{-1}([0,n]) \cup c_n^+$ which overlaps W (again see Lemma 2.9).

As in the quasifuchsian case, $\{\ell_{\rho_n}(c_n^+ \cup c_n^-)\}$ is bounded and $\{c_n^\pm\}$ has properties (3), (4) and (5) of Lemma 5.5.

Let $Y \subseteq S$ be an essential subsurface. If Y lies in a subsurface W which supports a conformal structure in ν_n^β. Then, as in the proof of inequality (5.1), Lemma 2.1 implies that

$$d_Y(\nu_n^\beta, c_n^\beta) \leq K$$

for large enough n as long as Y is not a component of **collar**(c_n^β). If a simple closed curve component p of ν_n^β intersects Y essentially, then $p \subset c_n^\beta$ and p is a closed curve without transversal in the base of the generalized marking $\mu(\nu_n^\beta)$ associated to ν_n^β (see section 2.2). Hence we have

$$d_Y(\nu_n^\beta, c_n^\beta) \leq 2.$$

Finally, if Y overlaps a subsurface W which is the base surface of a lamination component of ν_n^β, and $n \geq \alpha(d)$ for some $d \subset \partial Y$ that intersects W essentially, Theorem 2.8 then implies that

$$d_Y(\nu_n^\beta, c_n^\beta) \leq D.$$

Notice that in this last case we need $\partial Y \neq \emptyset$. We have proved that $\{c_n^\pm\}$ satisfies property (2). Since ν_n^β is never an ending lamination supported on all of S, ν_n^β contains either a closed curve or a conformal structure, so Property (1) holds as well. Lemma 5.5 then allows us to complete the proof in the case that ν_n^β is never an ending lamination supported on all of S.

To complete the proof, we consider the case where there exists $\beta_0 \in \{\pm\}$ such that for all n, $\nu_n^{\beta_0}$ is a lamination supported on all of S. Notice that in this case, Property (1) cannot hold, so we will need to again alter the construction somewhat.

If ν_n^β is not a lamination supported on all of S, then we choose c_n^β exactly as above. If ν_n^β is a lamination supported on all of S, then, by Minsky's Lipschitz Model Theorem [32], there exists L_0 and a tight geodesic g_n joining $\mu(\nu_n^+)$ to $\mu(\nu_n^-)$ such that for any vertex d of g_n, we have $\ell_{\rho_n}(d) \leq L_0$. Since $\{\nu_n^\pm\}$ bounds projections, there exists $K > 0$ and a vertex d_n of g_n, such that $d_S(d_n, \mu) \leq K$. Minsky's Lipschitz Model Theorem [32] again implies that there exists a pants decomposition c_n^β of S containing a vertex of g_n between d_n and $\mu(\nu_n^\beta)$ such that $\ell_{\rho_n}(c_n^\beta) \leq L_1$, and any curve in c_n^β lies above every curve in $\alpha^{-1}([0,n]) \cup e_n^-$ if $\beta = +$ and any curve in c_n^β lies below every curve in $\alpha^{-1}([0,n]) \cup c_n^+$ if $\beta = -$.

One then verifies properties (2)–(5) of Lemma 5.5 just as above. Property (1) was only used to prove condition (a), i.e. that every geodesic in $\mathscr{C}(S)$ joining c_n^+ to c_n^- passes through a fixed bounded set. However, in the case that $\nu_n^{\beta_0}$ is always a lamination supported on all of S, it follows directly from our construction and the hyperbolicity of the curve complex ([27]) that any geodesic joining c_n^+ to c_n^- passes within a uniformly bounded distance of μ. This completes the proof of Proposition 5.1 in our final case. □

6. Bounded projections implies convergence

In this section we prove that if a sequence of Kleinian surface groups admits a pair of sequences of multicurves of uniformly bounded length which bounds projections, then it has a convergent subsequence. We first handle the case where the sequence of multicurves does not have any combinatorial wrapped parabolics, and then handle the general case by applying an argument motivated by work of Kerckhoff and Thurston [22].

6.1. **In the absence of combinatorial wrapped parabolics.**

We recall that if a sequence $\{c_n^\pm\}$ of pairs of multicurves bounds projections and there are no combinatorial wrapped parabolics, then for any curve d and complete marking μ there exists $\beta(d)$ such that $\{m(c_n^{\beta(d)}, d, \mu)\}$ is eventually bounded.

Proposition 6.1. *Suppose that $\{\rho_n\}$ is a sequence in $AH(S)$ and there exists a sequence $\{c_n^\pm\}$ of pairs of multicurves such that $\{\ell_{\rho_n}(c_n^+ \cup c_n^-)\}$ is bounded and $\{c_n^\pm\}$ bounds projections and has no combinatorial wrapped parabolics. Then $\{\rho_n\}$ has a convergent subsequence.*

Remark: Notice that any bounded sequence in $QF(S)$ will admit bounded length multicurves which bound projections (any pair of filling pants decompositions will work). Therefore, we can only conclude that there exist a convergent subsequence.

Moreover, unlike in the end invariants case, a sequence of wrapped multicurves which bounds projections need not predict all the parabolics in the limit and need not predict which parabolics wrap. Notice that if $\{\rho_n\}$ converges and c^+ and c^- is any pair of filling multicurves, then the constant sequence $\{c_n^\pm = c^\pm\}$ will be a sequence of pairs of bounded length multicurves bounding projections. In this case, $\{c_n^\pm\}$ does not predict any parabolics or ending laminations.

Proof. We first show that, after passing to a subsequence $\{\rho_j\}$, there exists a fixed pants decomposition which has bounded length in all N_{ρ_j}.

Lemma 6.2. *Suppose that $\{\rho_n\}$ is a sequence in $AH(S)$ and consider a sequence $\{c_n^\pm\}$ of pairs of multicurves which bound projections without combinatorial wrapped parabolics. If $\{\ell_{\rho_n}(c_n^+ \cup c_n^-)\}$ is bounded, then there exists a subsequence $\{\rho_j\}$ and a pants decomposition r of S, so that $\{\ell_{\rho_j}(r)\}$ is a bounded sequence.*

Proof. By assumption, there is a bounded region $\mathscr{B}$ in $\mathscr{C}(S)$ such that any geodesic joining c_n^+ to c_n^- intersects $\mathscr{B}$. For all n, let b_n be a curve on the geodesic joining c_n^+ to c_n^- which is contained in $\mathscr{B}$. By Theorem 2.6, there exists D and L such that, for all n, there exists a curve $a_n \in \mathscr{C}(S)$ such that $d(a_n, b_n) \le D$ and $\ell_{\rho_n}(a_n) \le L$.

If $\{a_n\}$ admits a constant subsequence, then we pass to the appropriate subsequence of $\{\rho_n\}$ and the constant curve is the first curve in our pants decomposition r.

If not, by Lemma 2.3, there is a subsurface Y such that $d_Y(a_n, \mu)$ diverges. Since a_n is contained in a bounded region of $\mathscr{C}(S)$, Y is a proper subsurface of S. By assumption, there exists $\beta \in \{\pm\}$, so that $d_Y(c_n^\beta, \mu)$ is bounded, hence

$d_Y(c_n^\beta, a_n) \to \infty$. Then, by Theorem 2.5, $\ell_{\rho_n}(\partial Y) \to 0$. In this case, the components of ∂Y are the first curves in r.

We now assume that r is non-empty and not yet a pants decomposition. We apply a mild variation of the above argument to show that we can enlarge r. This will eventually complete the proof. Let W be a component of $S - r$ which is not a thrice-punctured sphere. Since r has uniformly bounded length, one may use Lemma 2.11 to find, for all n, a curve $b_n \in \mathscr{C}(W)$ so that $\ell_{\rho_n}(b_n)$ is uniformly bounded. By assumption, there exists $\beta \in \{\pm\}$ so that $d_W(c_n^\beta, \mu)$ is eventually bounded. Let $L \geq L_0$ be an upper bound for both $\{\ell_{\rho_n}(c_n^\beta)\}$ and $\{\ell_{\rho_n}(b_n)\}$ (where $L_0 = L_0(S)$ is the constant from Theorem 2.6). Theorem 2.6 implies that there exists $D = D(S,L)$ such that either $\mathrm{diam}(\pi_W(\mathscr{C}(\rho_n, L)) \leq D$ or $d_W(c_n^\beta, \mathscr{C}(W, L, \rho_n)) \leq D$ for all n (since $c_n^\beta \in \mathscr{C}(\rho, L)$). In the first case, $d_W(b_n, c_n^\beta) \leq D$, while in the second case there exists $a_n \in \mathscr{C}(W, L, \rho_n)$ such that $d_W(c_n^\beta, a_n) \leq D$. In the first case, we let $a_n = b_n$. Therefore, in either case, we have constructed a sequence $\{a_n\}$ in $\mathscr{C}(W)$ such that $\ell_{\rho_n}(a_n) \leq L$ and $d_W(c_n^\beta, a_n) \leq D$.

If $\{a_n\}$ admits a constant subsequence, then we pass to the appropriate subsequence of $\{\rho_n\}$ and add the constant curve to r. If not, by Lemma 2.3 there is a subsurface Y such that $d_Y(a_n, \mu)$ diverges. Since $\{d_W(a_n, \mu)\}$ is eventually bounded, Y is a proper subsurface of W. We can again argue, as in the third paragraph of the proof, that $d_Y(c_n^{\beta'}, a_n) \to \infty$ for some $\beta' \in \{\pm\}$. By Theorem 2.5, $\ell_{\rho_n}(\partial Y) \to 0$. In this case, we may add $\partial Y - \partial W$ to r. □

Next we construct, for every curve in r a transversal which has bounded length in all N_{ρ_j}, perhaps after passage to a further subsequence. By Lemma 2.11, there are bounded length pants decompositions r_j^+ and r_j^- in N_{ρ_j} containing c_j^+ and c_j^-, respectively. We may pass to a subsequence so that $r \cap r_j^+$ and $r \cap r_j^-$ are both constant. (Here, we use $r \cap r_j^\beta$ as shorthand for the collection of curves which lie in both r and r_j^β.)

Let d be a curve in r. There exists a choice of sign $\beta = \beta(d) \in \{\pm\}$ so that $m(c_j^\beta, d, \mu)$ is bounded for all j, perhaps after again passing to a subsequence. In particular, this implies that d does not lie in r_j^β (since d must intersect c_j^β if $m(c_j^\beta, d, \mu)$ is finite). Let $G = G(d)$ be the subsurface of $S - (r \cap r_j^\beta)$ which contains d.

Let $H_j = H_j(d)$ be a hierarchy in $\mathscr{C}(G)$ joining $r_j^\beta \cap G$ and $r \cap G$. Here we regard both $r_j^\beta \cap G$ and $r \cap G$ as markings without transversals. (Hierarchies are defined and discussed extensively in Masur-Minsky [28].)

Let $\sigma_j \in AH(G)$ be the unique Kleinian group so that $r_j^\beta \cap G$ is the collection of upward-pointing parabolic and $r \cap G$ is the collection of downward-pointing parabolics. Let $X_j = N_{\sigma_j} = \mathbf{H}^3/\sigma_j(\pi_1(G))$. (The hyperbolic manifold X_j is called a maximal cusp, see Keen-Maskit-Series [21] for a proof of the existence and uniqueness of X_j. The existence also follows from Thurston's Geometrization Theorem for pared manifolds, see Morgan [33].) Notice that $r_j^\beta \cap G$ and $r \cap G$ are the end invariants of X_j.

Let M_j be the model manifold associated to the hierarchy H_j. (The construction of a model manifold associated to a hierarchy is carried out in Minsky [32, Sec. 8].) The Bilipschitz Model Manifold Theorem [13] guarantees that there exists a bilipschitz homeomorphism $g_j : M_j \to X_j$.

The hierarchy H_j is a family of tight geodesics. The base tight geodesic lies in $\mathscr{C}(G)$ and joins $r_j^\beta \cap G$ to $r \cap G$. Theorem 2.7 implies that there is a uniform upper bound on the length $\ell_{\rho_j}(c)$ of any curve c which is contained in a vertex of the base tight geodesic. Then H_j is constructed iteratively by appending tight geodesics in curve complexes of subsurfaces of G which join vertices in previously added tight geodesics. Since this process terminates after a finite (bounded) number of steps, Theorem 2.7 implies that there is a uniform upper bound on the length $\ell_{\rho_j}(c)$ of any curve c contained in a vertex in the hierarchy H_j.

The model manifold M_j is constructed from *blocks* of two isometry types, one homeomorphic to the product of a one-holed torus and the interval and the other homeomorphic to the product of a four-holed sphere and the interval, *tubes*, which are isometric to Margulis regions in hyperbolic 3-manifolds, and a finite number of *boundary blocks*. Each block is associated to an edge of a geodesic in the curve complex of either a one-holed torus or a four-holed sphere. These geodesics are called 4-geodesics.

Let $\hat{M}_j$ be obtained from M_j by removing the tubes and the boundary blocks. So, $\hat{M}_j$ consists entirely of blocks. Since all the vertices have uniformly bounded length, the techniques of section 10 of Minsky [32] (in particular, see Steps 0–5) imply that there exists a K-Lipschitz map $h_j : \hat{M}_j \to N_{\rho_j}$ where K depends only on S and the uniform bound on the lengths of the curves in H_j obtained from Theorem 2.7.

Let $A_{d,j}$ be the intersection of $\hat{M}_j$ with $U(d)$, the tube in M_j associated to d. The annulus $A_{d,j}$ is made up of $s_j(d)+1$ bounded geometry annuli where $s_j(d)$ is the number of edges of 4-geodesics in H_j whose domains contain d in their boundary. The arguments in Theorem 9.11 of Minsky [32] imply that

$$s_j(d) \leq C\left(\sup_{d\in\partial Y, Y\neq \mathbf{collar}(d)} d_Y(r, r_j^\beta)\right)^a$$

for uniform constants C and a. However,

$$\sup_{d\in\partial Y, Y\neq \mathbf{collar}(d)} d_Y(r, r_j^\beta) \leq m(c_j^\beta, d, \mu) + \sup_{d\in\partial Y, Y\neq \mathbf{collar}(d)} d_Y(r, \mu).$$

The first term on the right hand side is uniformly bounded by assumption, while the second term is finite and independent of j. Therefore, $s_j(d)$ is bounded, which implies that the geometry of $A_{d,j}$ is uniformly bounded.

It follows that there is an essential curve $t_{d,j}$ of uniformly bounded length in $\partial\hat{M}_j$ which is disjoint from the boundaries of the annuli associated to components of $r \cap G - d$ and intersects $U(d)$ minimally, i.e. in two arcs if $U(d)$ separates the component of $G-(r\cap G)$ it is contained in and in one arc otherwise. The image $g_j(t_{d,j})$ in X_j is a curve, of uniformly bounded length, which lies above the cusp associated to d. Theorem 2.8 then implies that $d_Y(t_{j,d},r_j^\beta)$ is uniformly bounded when $d\subset\partial Y$. Since, $m(c_j^\beta,\mu)$ is uniformly bounded and

$$|d_Y(c_j^\beta,\mu)-d_Y(r_j^\beta,\mu)|\leq 1,$$

we see that $d_Y(t_{j,d},\mu)$ is uniformly bounded for any subsurface $Y\subset S$ whose boundary contains d. Since any two curves which are disjoint from $r\cap G-d$ and intersect d minimally differ, up to homotopy, by a power of a Dehn twist in $U(d)$, there are only finitely many possibilities for $t_{j,d}$. Therefore, we may pass to a subsequence so that $t_{j,d}=t_d$ for a fixed curve t_d. The length $\ell_{\rho_j}(t_d)$ is uniformly bounded, since $h_j(t_d)$ is a bounded length representative of t_d in N_{ρ_j}.

We have found a pants decomposition r and a system of transversals $\{t_d\}_{d\in r}$ such that all curves in r and their transversals have uniformly bounded length in $\{N_{\rho_j}\}$. It then follows from Thurston's Double Limit Theorem [38, 35] that $\{\rho_j\}$ has a convergent subsequence. □

Remark: With a little more care, one may use this same argument to find a surface in N_{ρ_j}, for all large enough j, where r and $\{t_d\}_{d\in r}$ have uniformly bounded length. One can then verify convergence up to subsequence more directly.

6.2. **The general case.** We now use ideas based on work of Kerckhoff and Thurston [22] to handle the general case.

Proposition 6.3. *Suppose that $\{\rho_n\}$ is a sequence in $AH(S)$ and there exists a sequence of pairs, $\{c_n^\pm\}$, of multicurves such that $\{\ell_{\rho_n}(c_n^+\cup c_n^-)\}$ is bounded and $\{c_n^\pm\}$ bounds projections. Then $\{\rho_n\}$ has a convergent subsequence.*

Proof. Let q be the set of combinatorial wrapped parabolics for $\{c_n^\pm\}$. We recall that $d\in q$ if and only if $\{m^{na}(c_n^+,d,\mu)\}$ and $\{m^{na}(c_n^-,d,\mu)\}$ are both eventually bounded and there exists $w=w(d)\in\mathbf{Z}$ and a sequence $\{s_n=s_n(d)\}\subset\mathbb{Z}$ such that $\lim|s_n|=\infty$ and both $\{d_Y((D_Y^{s_nw}(c_n^+),\mu)\}$ and $\{d_Y(D_Y^{s_n(w-1)}(c_n^-),\mu)\}$ are eventually bounded when $Y=\mathbf{collar}(d)$.

Notice that if q is empty, then Proposition 6.3 follows from Proposition 6.1. We first observe that q is a multicurve.

Lemma 6.4. *The set q of combinatorial wrapping parabolics is a multicurve.*

Proof. Suppose that q contains intersecting curves c and d, and let $Y=\mathbf{collar}(c)$ and $Z=\mathbf{collar}(d)$. Lemma 2.2 then implies that

$$\min\{d_Y(\partial Z,c_n^+),d_Z(\partial Y,c_n^+)\}\leq 10$$

which contradicts the fact that both $d_Y(c_n^+,\mu)\to\infty$ and $d_Z(c_n^+,\mu)\to\infty$. □

Let $Q=\bigcup_{q_i\in q}Q_i=\mathbf{collar}(q_i)$ be a regular neighborhood of q and consider the diffeomorphisms

$$\Phi_n^+=\Pi_{q_i\in q}D_{Q_i}^{s_n(q_i)w(q_i)}\ \text{ and }\ \Phi_n^-=\Pi_{q_i\in q}D_{Q_i}^{s_n(q_i)(w(q_i)-1)}$$

where D_{Q_i} is the right Dehn twist about the annulus Q_i.

Lemma 6.5. *The pairs of sequences $\{\Phi_n^+(c_n^\pm)\}$ and $\{\Phi_n^-(c_n^\pm)\}$ both bound projections and have no combinatorial wrapped parabolics.*

Proof. We first prove that $\{\Phi_n^+(c_n^\pm)\}$ bounds projections.

Let d be a curve in q. Since $\{c_n^\pm\}$ bounds projections, d lies a uniformly bounded distance from any geodesic joining c_n^+ to c_n^-. Notice that if $c \in \mathscr{C}(S)$, then $d_S(d,\Phi_n^+(c)) = d_S(d,c)$. Since any geodesic joining $\Phi_n^+(c_n^+)$ to $\Phi_n^+(c_n^-)$ is the image under Φ_n^+ of a geodesic joining c_n^+ to c_n^-, it follows that d also lies a uniformly bounded distance from any geodesic joining $\Phi_n^+(c_n^+)$ to $\Phi_n^+(c_n^-)$. Hence the pair of sequence $\{\Phi_n^+(c_n^+)\}$ and $\{\Phi_n^+(c_n^-)\}$ satisfies condition (a) in the definition of bounding projections.

Let $d \subset S$ be a simple closed curve which is not a component of q. If d does not cross q then $m(c_n^\pm, d, \mu) = m(\Phi_n^+(c_n^\pm), d, \mu)$ for all n. Since $\{c_n^\pm\}$ bounds projections and d is not a combinatorial wrapping parabolic, it follows that there exists $\beta \in \{\pm\}$ such that $\{m(\Phi_n^+(c_n^\beta), d, \mu)\}$ is eventually bounded.

If d crosses a component q_i of q, it follows from the definition of $\Phi_n^\pm$ that $d_{Q_i}(d,\Phi_n^+(c_n^-)) \longrightarrow \infty$ where Q_i is the collar neighborhood of q_i. Lemma 2.2 then implies that if n is large enough, then $d_Y(q_i,\Phi_n^+(c_n^-)) \leq 4$ for any subsurface Y whose boundary contains d. Thus, again if n is large enough, by Lemma 2.1,

$$d_Y(\mu,\Phi_n^+(c_n^-)) \leq d_Y(\mu,q_i) + d_Y(q_i,\Phi_n^+(c_n^-)) \leq 1 + 2i(q_i,\mu) + 4 = 5 + 2i(q_i,\mu)$$

for any subsurface Y whose boundary contains d. Therefore, $\{m(\Phi_n^+(c_n^-), d, \mu)\}$ is eventually bounded.

If $d = q_i$ is a component of Q, then $m^{na}(c_n^+, q_i, \mu) = m^{na}(\Phi_n^+(c_n^+), q_i, \mu)$ for all n, so $\{m^{na}(\Phi_n^+(c_n^+), q_i, \mu)\}$ is eventually bounded. By definition of Φ_n^+, $\{d_{Q_i}(\Phi_n^+(c_n^+), \mu)$ is eventually bounded. Therefore, $\{m(\Phi_n^+(c_n^+), q_i, \mu)\}$ is eventually bounded

We have proved that for any simple closed curve $d \subset S$ there is β such that $m(\Phi_n^+(c_n^\beta), d, \mu)$ is eventually bounded. This completes the proof that the pair $\{\Phi_n^+(c_n^\pm)\}$ bounds projections without combinatorial wrapped parabolics.

The proof that the sequences of pairs $\{\Phi_n^-(c_n^\pm)\}$ bounds projections without combinatorial wrapped parabolics is analogous. □

For each n, consider the representations

$$\rho_n^+ = \rho_n \circ (\Phi_n^+)_*^{-1} \quad \text{and} \quad \rho_n^- = \rho_n \circ (\Phi_n^-)_*^{-1}.$$

By construction, the sequences $\{\ell_{\rho_n^\beta}(\Phi_n^\beta(c_n^\pm))\} = \{\ell_{\rho_n}(c_n^\pm)\}$ are uniformly bounded for any $\beta \in \{\pm\}$. Lemma 6.5 implies that $\{\Phi_n^+(c_n^\pm)\}$ and $\{\Phi_n^-(c_n^\pm)\}$ both bound projections and have no combinatorial wrapped parabolics, so Proposition 6.1 implies that we may pass to a subsequence so that both $\{\rho_n^+\}$ and $\{\rho_n^-\}$ converge to discrete, faithful representations ρ^+ and ρ^-.

Extend q to a pants decomposition p of S. If $d \in p$, then $\ell_{\rho_n}(d) = \ell_{\rho_n^+}(d)$ for all n, so $\{\ell_{\rho_n}(d)\}$ is bounded. Let $\hat{p}$ be a maximal collection of transversals to the elements of p (i.e. each element of $\hat{p}$ intersects exactly one element of p and does so minimally). If $t \in \hat{p}$ is a transversal to an element of $p-q$, then again $\ell_{\rho_n}(t) = \ell_{\rho_n^+}(t)$ for all n, so $\{\ell_{\rho_n}(t)\}$ is bounded

Lemma 6.6. *If $t \in \hat{p}$ is a transversal to an element d of q, then $\{\ell_{\rho_n}(t)\}$ is bounded.*

Proof. We show that any subsequence of $\{\rho_n\}$ contains a further subsequence such that $\{\rho_n(t)\}$ converges. Our result then follows immediately.

We first pass to a subsequence, and fix a specific representative in each conjugacy class, so that $\{\rho_n^+ = \rho_n \circ (\Phi_n^+)_*^{-1}\}$ converges as a sequence of representations into $\mathrm{PSL}(2,\mathbb{C})$. (The existence of such a subsequence follows from Lemma 6.5 and Proposition 6.1.) Since Φ_n^+ and Φ_n^- restrict to the identity on $S-Q$, and $\{\rho_n^-\}$ has a convergent subsequence in $AH(S)$ (again by Lemma 6.5 and Proposition 6.1), we may pass to a further subsequence so that $\{\rho_n^-\}$ also converges as a sequence of representations into $\mathrm{PSL}(2,\mathbb{C})$.

Let us first consider the case where t intersects d exactly once. Then, with an appropriate choice of basepoint for $\pi_1(S)$, we have

$$\rho_n^-(t) = \rho_n(d^{(w(d)-1)s_n}t) = \rho_n^+(d^{-s_n}t),$$

so $\rho_n^+(d^{-s_n}) = \rho_n^-(t)\rho_n^+(t)^{-1}$. Since $\{\rho_n^-(t)\}$ and $\{\rho_n^+(t)\}$ both converge we immediately conclude that $\{\rho_n^+(d^{s_n}) = \rho_n(d^{s_n})\}$ and $\{\rho_n(t) = \rho_n(d^{-w(d)s_n})\rho_n^+(t)\}$ converge.

In the slightly more complicated second case where t intersects d twice, we argue by contradiction. We first homotope t so that the two points of $t \cap d$ coincide. Then t is the concatenation of two loops a and b which are freely homotopic to curves that are disjoint from d and $\rho_n(t) = \rho_n(ab)$. With an appropriate choice of basepoint for $\pi_1(S)$, we have

$$\rho_n(a) = \rho_n^+(a) = \rho_n^-(a), \quad \rho_n(d) = \rho_n^+(d) = \rho_n^-(d),$$

and

$$\rho_n^-(b) = \rho_n(d^{(w(d)-1)s_n}bd^{-(w(d)-1)s_n}) = \rho_n^+(d^{-s_n}bd^{s_n}).$$

Suppose that $\{\rho_n(d^{s_n}) = \rho_n^+(d^{s_n}\}$ exits every compact subset of $\mathrm{PSL}(2,\mathbb{C})$ and pick $p \in \mathbb{H}^3$. Since the fixed points of $\rho_n^+(d)$ and $\rho_n^+(b)$ converge to distinct sets (i.e. the fixed points of $\rho^+(d)$ and $\rho^+(b)$), $\rho_n^+(d^{s_n})(p)$ converges to a point in $\partial\mathbb{H}^3$ disjoint from the fixed point set of $\rho^+(b)$. It follows that

$$d(\rho_n^+(bd^{s_n})(p), \rho_n^+(d^{s_n})(p)) \to \infty.$$

Applying $\rho_n^+(d^{-s_n})$ to each term we see that

$$d(\rho_n^+(d^{-s_n}bd^{s_n})(p), p) \to \infty,$$

which contradicts the fact that $\{\rho_n^-(b) = \rho_n^+(d^{-s_n}bd^{s_n})\}$ converges. Therefore, a subsequence of $\{\rho_n(d^{s_n})\}$ converges. It follows that, with the same subsequence, $\{\rho_n(b) = \rho_n(d^{-w(d)s_n})\rho_n^+(b)\rho_n(d^{w(d)s_n})\}$ and $\{\rho_n(t) = \rho_n(ab)\}$ both converge. (For a related argument see Anderson-Lecuire [4, Claim 7.1].) This completes the proof. □

We have exhibited a pants decomposition and a complete collection of transversals all of whose images under ρ_n have bounded length. Therefore, Thurston's Double Limit Theorem [38, 35] again implies that $\{\rho_n\}$ has a convergent subsequence. □

7. CONCLUSION

We will now assemble the previous results to establish Theorems 1.1, 1.2 and 1.3. Let S be a compact, orientable surface and let $\{\rho_n\}$ be a sequence in $AH(S)$ with end invariants $\{\nu_n^\pm\}$.

Proof of Theorem 1.1: If $\{\nu_n^\pm\}$ has a subsequence $\{\nu_j^\pm\}$ which bounds projections, then Proposition 5.1 implies that there exists a further subsequence, still called $\{\rho_j\}$, and a sequence $\{c_j^\pm\}$ of pairs of multi-curves such that $\{\ell_{\rho_j}(c_j^+ \cup c_j^-)\}$ is bounded and $\{c_j^\pm\}$ bounds projections. Theorem 6.3 then implies that $\{\rho_j\}$, and hence $\{\rho_n\}$, has a convergent subsequence. On the other hand, if $\{\rho_n\}$ has a convergent sequence, it follows immediately from Theorem 4.1 that some subsequence of $\{\nu_n^\pm\}$ bounds projections. □

Theorem 1.2 is precisely the second part of Theorem 4.1.

Proof of Theorem 1.3: Theorem 6.3 implies that if there exists a sequence $\{c_n^\pm\}$ of pairs of multi-curves such that $\{\ell_{\rho_n}(c_n^+ \cup c_n^-)\}$ is bounded and $\{c_n^\pm\}$ bounds projections, then $\{\rho_n\}$ has a convergent subsequence. On the other hand, if $\{\rho_n\}$ has a convergent subsequence $\{\rho_j\}$, then we may simply pick any filling pair $c^\pm$ of multi-curves and set $c_j^\pm = c^\pm$ for all j. Then, since $\{\rho_j\}$ is convergent, $\{\ell_{\rho_j}(c_j^+ \cup c_j^-)\}$ is bounded and $\{c_j^\pm\}$ bounds projections □

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