CONVERGENCE OF FREELY DECOMPOSABLE KLEINIAN GROUPS

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Abstract. We consider a compact orientable hyperbolic 3-manifold with a compressible boundary. Suppose that we are given a sequence of geometrically finite hyperbolic metrics whose conformal boundary structures at infinity diverge to a projective lamination. We prove that if this limit projective lamination is doubly incompressible, then the sequence has compact closure in the deformation space. As a consequence we generalise Thurston's double limit theorem and solve his conjecture on convergence of function groups affirmatively.

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1. Introduction

It is one of the most important topics in the theory of Kleinian group to study the topological structure of their deformation spaces. The quasi-conformal deformation space of a geometrically finite Kleinian group G is fairly well understood by virtue of the work of Ahlfors, Bers, Kra, Marden and Sullivan, and others. To put it more concretely, for a geometrically finite Kleinian group G, it is known that there is a ramified covering map from the Teichmüller space of Ω_G/G to the quasi-conformal deformation space of G, where Ω_G denotes the region of discontinuity of G. On the other hand, in general it is difficult to understand how these coordinates on Teichmüller space relate to the full deformation space. In other words, it is a challenging task to determine which divergence sequence in Teichmüller space correspond to convergent sequences of Kleinian groups.

The first example of such a sufficient condition for convergence is the result of Bers in [Ber], which shows that the space of quasi-Fuchsian groups lying on a Bers slice is relatively compact. On the other hand, in the process of proving the uniformisation theorem for Haken manifolds, Thurston proved the double limit theorem for quasi-Fuchsian groups and the compactness of deformation spaces for acylindrical manifolds, in [Th1] and [Th2] respectively. These are generalised to give a convergence theorem for general freely indecomposable Kleinian groups in Ohshika [Oh1] and [Oh2]. The convergence in the deformation spaces for freely decomposable groups is more complicated and is harder to understand.

In [ThB], Thurston asked how one might generalise the double limit theorem to the setting of Schottky groups. This question was made into a more concrete conjecture using the notion of Masur domain, and then was generalised to function groups. Masur introduced in [Ma] an open set in the projective lamination space of the boundary of a handlebody on which the mapping class group of the handlebody acts properly discontinuously. This open set is what we call the Masur domain nowadays. This notion is generalised by Otal [Ot1] to the exterior boundary of a compression body. Thurston's conjecture is paraphrased as follows: For a sequence in Teichmüller space converging in the Thurston compactification to a projective lamination lying in the Masur domain of the exterior boundary of a compression body M, the corresponding sequence of convex cocompact representations in $AH(M)$ has a convergent subsequence. Otal in [Ot2] first proved that Thurston's conjecture is true for rank-2 Schottky space provided that the limit lamination is arational, that is, any component of its complement is simply connected. Canary in [Ca] proved the conjecture for some special sequences in Schottky space. Ohshika in [Oh4] proved the conjecture for function groups which are isomorphic to the free products of two surface groups under the same assumption that the limit lamination is arational. The strongest result in this direction under the same assumption on the limit lamination was given by

Kleineidam and Souto in [KlS] without any other assumption on compression bodies. Our main result, Theorem 1 yields a proof of this conjecture of Thurston in full generality without any extra assumption and generalises it to a slightly larger set than the Masur domain.

We need to introduce some notions and notations to state our main theorem. Consider a compact irreducible atoroidal 3-manifold M with boundary. By Thurston's uniformisation theorem for atoroidal Haken manifolds, there is a representation $\rho_0 : \pi_1(M) \to \mathrm{Isom}(\mathbb{H}^3)$ such that $\mathbb{H}^3/\rho_0(\pi_1(M))$ is homeomorphic to $\mathrm{Int}(M)$ by a homeomorphism that induces ρ. Such a representation is said to *uniformise* M. Any quasi-conformal deformation of ρ_0 also uniformises M. By the Ahlfors-Bers theory, when ρ_0 is convex-cocompact, the space $QH(\rho_0)$ of quasi-conformal deformations of ρ_0 up to conjugacy by elements of $\mathrm{Isom}(\mathbb{H}^3)$ is parametrised by the Teichmüller space of the boundary of M. More precisely, there is a (possibly ramified) covering map, called the *Ahlfors-Bers map* $\mathcal{T}(\partial M) \to QH(\rho_0)$ whose covering transformation group is the group of isotopy classes of diffeomorphisms of M which are homotopic to the identity.

The space $QH(\rho_0)$ is a subspace of the *deformation space* $AH(M)$. This deformation space $AH(M)$ is the space of discrete faithful representations $\rho : \pi_1(M) \to \mathrm{Isom}(\mathbb{H}^3)$ up to conjugacy by elements of $PSL_2(\mathbb{C})$. It is endowed with the quotient of the compact-open topology which is also called the *algebraic topology*. In the main theorem, we shall consider sequences of representations given by sequences in the Teichmüller space whose images under the Ahlfors-Bers map diverge in $QH(\rho_0)$ and give a sufficient condition for their convergence in $AH(M)$.

Thurston introduced in [Th3] the notion of *doubly incompressible curves*. This can be extended to measured geodesic laminations in the following way.

We say that a measured geodesic lamination $\lambda \in \mathcal{ML}(\partial M)$ is *doubly incompressible* if and only if there exists $\eta > 0$ such that $i(\lambda, \partial E) > \eta$ for any essential annulus or disc E in M. We denote by $\mathcal{D}(M) \subset \mathcal{ML}(\partial M)$ the set of doubly incompressible measured geodesic laminations and by $\mathcal{PD}(M)$ its projection in the projective lamination space $\mathcal{PML}(\partial M)$. It is not hard to see that $\mathcal{D}(M)$ contains the Masur domain (see [Le2]). Our main theorem is the following.

Theorem 1. *Let M be a compact orientable irreducible atoroidal 3-manifold with boundary and $\rho_0 : \pi_1(M) \to PSL(2, \mathbb{C})$ a convex cocompact representation that uniformises M. Let (m_n) be a sequence in the Teichmüller space $\mathcal{T}(\partial M)$ that converges in the Thurston compactification to a projective measured lamination $[\lambda]$ contained in $\mathcal{PD}(M)$. Let $q : \mathcal{T}(\partial M) \to QH(\rho_0)$ be the Ahlfors-Bers map, and suppose that $(\rho_n : \pi_1(M) \to G_n \subset PSL(2, \mathbb{C}))$ is a sequence of discrete faithful representations corresponding to $q(m_n)$. Then (ρ_n) has an algebraically convergent subsequence in $AH(M)$.*

It should be noted that our result here is closely related to the Bers-Thurston density conjecture. This conjecture states that every finitely generated Kleinian group is contained in the boundary of the quasi-conformal deformation space of geometrically finite Kleinian groups without rank-1 maximal parabolic subgroups. A special case has been proved by Bromberg and Brock-Bromberg [Brom, BB] using cone manifold deformation theory. Bromberg and Souto announced a complete prof of the general case using this approach and, in particular, avoiding an appeal to the ending lamination theorem ([BS]). The general case can be proved by combining the resolution of the tameness conjecture by Agol and Calegari-Gabai, the ending lamination conjecture by Brock-Canary-Minsky, and from some convergence theorems due to Thurston, Ohshika, Kleineidam-Souto and Lecuire, together with some topological argument due to Ohshika and Namazi-Souto(See [Ag], [CG], [Th3], [Oh4], [KlS], [Le2], [Min], [Oh5] and [NaS]).

Theorem 1 is a corollary of Theorem 2 below by the following argument. By Theorems of Thurston [Th2] and Canary [Ca], the convergence of (m_n) to $[\lambda]$ implies that there is a sequence of weighted multi-curves $(\lambda_n \in \mathcal{ML}(\partial M))$ such that $l_{\rho_n}(\lambda_n)$ tends to 0 and that the sequence (λ_n) converges in $\mathcal{ML}(\partial M)$ to a measured geodesic lamination whose projective class is $[\lambda]$. Since λ lies in $\mathcal{D}(M)$, Theorem 1 is derived from the following theorem, whose proof occupies the main part of this paper.

Theorem 2. *Let $(\rho_n : \pi_1(M) \to \mathrm{Isom}(\mathbb{H}^3))$ be a sequence of convex cocompact representations that uniformise M and let $(\lambda_n) \subset \mathcal{ML}(\partial M)$ be a sequence of measured geodesic laminations such that $(l_{\rho_n}(\lambda_n))$ is a bounded sequence and that (λ_n) converges in $\mathcal{ML}(\partial M)$ to a measured geodesic lamination $\lambda \in \mathcal{D}(M)$. Then the sequence (ρ_n) has a compact closure in $AH(M)$, namely, any subsequence contains an algebraically convergent subsequence.*

We prove Theorem 2 by contradiction using the following arguments. We can first assume that λ_n is a weighted simple closed curve since the set of weighted simple closed curves is dense in $\mathcal{ML}(S)$. Assuming that no subsequence of (ρ_n) converges, we use Culler-Morgan-Shalen compactification of the character variety ([MoS1]) and ideas of Otal [Ot3] to construct a sequence of train tracks τ_n carrying λ_n and maps $f_n : \tau_n \to \mathbb{H}^3/\rho_n(\pi_1(M))$ so that some parts of τ_n are mapped to long quasi-geodesic paths while the rest is mapped to relatively short paths. Then we construct a simplicial annulus joining $f_n(\lambda_n)$ to its geodesic representative $\lambda_n^* \subset M_n = \mathbb{H}^3/\rho_n(\pi_1(M))$. By Gauss-Bonnet Formula, this annulus is not large enough to cover the difference in length between $f_n(\lambda_n)$ and λ_n^*. It follows that $f_n(\lambda_n)$ nearly backtracks along some long paths. We use this backtracking to construct a sequence of annuli whose boundary converges to a sublamination of λ. This contradicts the double incompressibility of λ.

We state Theorem 2 under the assumption that ρ_n is convex cocompact but it should hold without this assumption, namely when ρ_n is discrete and faithful. The presence of parabolics, in particular rank 2 cusps, would add extra technicalities to the proof that we wished to avoid. The presence of geometrically infinite ends would require very little change. In the same spirit, we state Theorem 1 under the assumption that ρ_n is convex cocompact but it should hold under the assumption that it is geometrically finite and minimally parabolic.

Plan of the paper:

In Section 2, we explain background materials and quote necessary results, some with proofs.

In Sections 3 and 4 we explain the construction of τ_n and h_n. In Section 3, we show that for a doubly incompressible measured lamination λ and a small minimal action of $\pi_1(M)$ on $\mathbb{R}$-tree $\mathcal{T}$, each component of λ is either realised in $\mathcal{T}$, or is carried by train tracks with arbitrarily short branches in the following sense: there is a sequence of train tracks θ_i minimally carrying the component, such that the branches of θ_i are mapped to geodesic segments in $\mathcal{T}$ with lengths going to zero. This is proved in Lemma 3.1.

Using this result, we construct τ_n and f_n in Section 4. By a theorem of Morgan and Shalen ([MoS1]), if (ρ_n) does not have a compact closure in $AH(M)$, a subsequence of (ρ_n) tends to a minimal small action of $\pi_1(M)$ on an $\mathbb{R}$-tree $\mathcal{T}$. Let L_{rec} be the union of the recurrent leaves of the Hausdorff limit L_∞ of $|\lambda_n|$ where the λ_n are measured laminations with $(l_{\rho_n}(\lambda_n))$ bounded. We construct a sequence of train tracks $\tau_n = \tau^1 \cup \tau_n^2 \cup \tau_n^3$ where τ^1 carries realised components of L_{rec}, τ_n^2 carries non-realised components of L_{rec} and τ_n^3 carries $L_\infty - L_{rec}$ with small weight with respect to λ_i. We also construct a sequence of ρ_n-equivariant maps $\hat{f}_n$ from the universal cover $\hat{\tau}_n$ of τ_n to $\mathbb{H}^3$ which map the branches of $\hat{\tau}_1$ to long geodesic segments and the branches of $\hat{\tau}_n^2$ to comparatively short ones. This is proved in Lemma 4.1.

In Section 5 we show that there are long parts of $f_n(\lambda_n)$ which nearly backtrack. We approximate λ_n by weighted simple closed curves c_n. By hypothesis, the geodesic representatives $c_n^* \subset M_n = \mathbb{H}^3/\rho_n(\pi_1(M))$ of c_n have bounded lengths (taking the weight also into account). On the other hand, by construction, $f_n(c_n)$ gets infinitely long with n. We construct an annulus between $f_n(c_n)$ and c_n^* whose area is controlled. In this annulus, it follows from the difference in length between $f_n(c_n)$ and c_n^* that, for large enough n, the path $f_n(c_n)$ has a number of long segments in which it comes back nearly parallel to itself. This provides us with some long and thin strips connecting two segments of $f_n(c_n)$.

In Section 6, we explain how these strips give rise to discs or annuli whose boundaries converge in the Hausdorff topology to a geodesic lamination which does not intersect λ transversely.

In Section 7 we deduce Theorem 2 and 1 from the results in the preceding sections.

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2. Preliminaries

2.1. **Deformation space.** Let G be a finitely generated torsion-free Kleinian group, namely a (torsion-free and finitely generated) discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$. The group G is *convex cocompact* if there is a compact subset $C \subset \mathbb{H}^3/G$ that contains all the closed geodesics. Denote by Ω_G the domain of discontinuity for the action of G on $\hat{\mathbb{C}}$. The group G is convex cocompact if and only $\mathbb{H}^3/G \sqcup \Omega_G/G$ is compact.

Let M be a compact orientable irreducible atoroidal manifold. Let $AH(M)$ denote the set of faithful discrete representations from $\pi_1(M)$ to $PSL(2,\mathbb{C})$ modulo conjugacy. We endow $AH(M)$ with the topology induced from the representation space. The subspace of $AH(M)$ consisting of convex cocompact representations is denoted by $CC(M)$. This space $CC(M)$ is not empty if and only if ∂M contains no tori and it may contain several connected components. The component consisting of representations ρ for which there is a homeomorphism from $\mathrm{Int}(M)$ to $\mathbb{H}^3/\rho(\pi_1(M))$ that induces ρ is denoted by $CC_0(M)$. For such representations, the homeomorphism $\mathrm{Int}(M) \to \mathbb{H}^3/\rho(\pi_1(M))$ extends to a homeomorphism $M \to \mathbb{H}^3/\rho(\pi_1(M)) \sqcup \Omega_\rho/\rho(\pi_1(M))$. This produces a natural identification of $\Omega_\rho/\rho(\pi_1(M))$ with ∂M. Notice that the homeomorphism $M \to \mathbb{H}^3/\rho(\pi_1(M)) \sqcup \Omega_\rho/\rho(\pi_1(M))$ is well defined up to composition by an element of $\mathrm{Mod}_0(M)$ (the set of diffeomorphisms of M that are homotopic to the identity map).

For a Kleinian group G, if there is a quasi-conformal automorphism f of S^2_∞ such that fGf^{-1} is again a Kleinian group, then this group fGf^{-1} is said to be a *quasi-conformal deformation* of G. We denote by $QH(G)$ the space of quasi conformal deformations of G up to conjugacy. By the theory of Ahlfors-Bers, there is a ramified covering map from $\mathcal{T}(\Omega_G/G)$ to the space of quasi-conformal deformations of G modulo conjugacy.

Given a representation $\rho_0 \in CC_0(M)$, we have $QH(\rho_0(\pi_1(M)) = CC_0(M)$. As we have seen, for any $\rho \in CC_0(M)$, there is a natural identification of $\Omega_\rho/\rho(\pi_1(M))$ with ∂M. Thus the theory of Ahlfors-Bers provides us with a covering map $\mathcal{T}(\partial M) \to CC_0(M)$ which we call the *Ahlfors-Bers map*.

2.2. **$\mathbb{R}$-trees.** An $\mathbb{R}$-tree $\mathcal{T}$ is a geodesic metric space in which any two points x, y can be joined by a unique simple arc. Let G be a group acting by isometries on an $\mathbb{R}$-tree $\mathcal{T}$. The action is *minimal* if there is no proper invariant subtree and *small* if the stabilizer of any non-degenerate arc is virtually Abelian.

Morgan and Shalen [MoS1] made use of $\mathbb{R}$-trees to compactify deformation spaces. They used algebraic methods involving valuations, while the same result has been obtained by Paulin [Pa] and Bestvina [Bes] using a more geometrical approach. In this paper we shall adopt the point of view of Kapovich-Leeb [KaL] (see also [Ka, chapters 9 and 10]). Let

$(\rho_n \subset GF_0(M,P))$ be a sequence of representations such that no subsequence of (ρ_n) converges algebraically. Let $\Gamma \subset \pi_1(M)$ be a set of generators and let $\tilde{x}_n \subset \mathbb{H}^3$ be a point realising the minimum ϵ_n^{-1} on $\mathbb{H}^3$ of the function $\max\{d(\tilde{x}, \rho_n(a)(\tilde{x})), a \in \Gamma\}$ (see for example [Pa, Lemma 6.5] for the existence of such a point). Since no subsequence of (ρ_n) converges algebraically, (ϵ_n^{-1}) tends to ∞. Choose a non-principal ultra-filter ω and denote by $\epsilon_n\mathbb{H}^3$ the hyperbolic space $\mathbb{H}^3$ with the hyperbolic metric rescaled by ϵ_n. The ultra-limit $(X_\omega, x) = \omega - \lim(\epsilon_n\mathbb{H}^3, x_n)$ of the sequence of rescaled spaces is defined as follows. Let $\Pi_n(\epsilon_n\mathbb{H}^3)$ be the infinite product of the spaces $(\epsilon_n\mathbb{H}^3)$. We define a function d_ω on $\Pi_n(\epsilon_n\mathbb{H}^3)$ by setting

$$d_\omega(y,z) = \omega - \lim d_{\epsilon_n\mathbb{H}^3}(\tilde{y}_n, \tilde{z}_n)$$

for any two points $y = (\tilde{y}_n)$ and $z = (\tilde{z}_n)$ lying in $\Pi_n(\epsilon_n\mathbb{H}^3)$.

This function d_ω is a pseudo-distance in $\Pi_n(\epsilon_n\mathbb{H}^3)$ with values in $[0,\infty]$ and we set $(X_\omega, d_\omega) = (\Pi_n(\epsilon_n\mathbb{H}^3), d_\omega)/\sim$ where we identify points with zero d_ω-distance. Let $x = (\tilde{x}_n)$ denote the sequence of points $\tilde{x}_n$ defined above. The metric space (X_ω, x) is the set of points of (X_ω) with a finite distance from x. This metric space is an $\mathbb{R}$-tree (cf. [KaL]). The action of $\rho_n(\pi_1(M))$ on $\epsilon_n\mathbb{H}^3$ gives rise to an action of $\pi_1(M)$ on (X_ω, x) by isometries. This action is small (cf. [KaL]). Let $\mathcal{T}$ be the minimal invariant subtree of X_ω under this action. We say that (ρ_n) tends to the action of $\pi_1(M)$ on $\mathcal{T}$ with respect to ω. For $c \in \pi_1(M)$ let us denote by $\delta_{\mathcal{T}}(c)$ the minimal translation distance of c on $\mathcal{T}$. Then we have $\delta_{\mathcal{T}}(c) = \omega - \lim \epsilon_n l_{\rho_n}(c)$, where $l_{\rho_n}(c)$ is the length in $\mathbb{H}^3/\rho_n(\pi_1(M))$ of the closed geodesic in the free homotopy class of c.

2.3. **Geodesic laminations.** A *geodesic lamination* L on a complete hyperbolic surface S is a compact set which is a disjoint union of complete embedded geodesics called *leaves*. It is known that this definition is independent of a chosen hyperbolic metric on S. The details can be found in [Ot3]. For a connected geodesic lamination L which is not a simple closed curve we denote by $\bar{S}(L)$ the smallest subsurface of S with compact geodesic boundary containing L. Inside $\bar{S}(L)$ there are finitely many closed geodesics (including the components of $\partial\bar{S}(L)$) disjoint from L. These closed geodesics do not intersect each other (cf. [Le1] page 99) and we denote by $\partial'\bar{S}(L) \supset \partial\bar{S}(L)$ their disjoint union. For example if a component of $\bar{S}(L)\setminus L$ is an annulus, its core curve is contained in $\partial'\bar{S}(L)$ but not in $\partial\bar{S}(L)$. Removing a small open tubular neighbourhood of $\partial'\bar{S}(L)$ from $\bar{S}(L)$ we get a compact surface $S(L)$. We call $S(L)$ the *surface embraced* by the geodesic lamination L and $\partial'\bar{S}(L)$ the *effective boundary* of $S(L)$. If L is a simple closed curve, we define $S(L)$ to be an annular neighbourhood of L and we take $\partial'\bar{S}(L) = L$. When L is not connected, $S(L)$ is the disjoint union of the surfaces embraced by the connected components of L.

We say that a geodesic measured lamination L *crosses* another geodesic lamination L' if *at least* one leaf of L intersects a leaf of L' transversely.

A *measured geodesic lamination* λ is a geodesic lamination $|\lambda|$ together with a transverse measure of full support. We denote by $\mathcal{ML}(S)$ the space of measured geodesic laminations on S endowed with the weak-$*$ topology. To simplify the notations, we write $\mathcal{ML}(\partial M)$ instead of $\mathcal{ML}(\partial_{\chi<0} M)$ for a compact 3-manifold M with boundary. The projective lamination space $\mathcal{PML}(\partial M)$ is defined to be $(\mathcal{ML}(\partial M) - \{0\})/\mathbb{R}_+^*$ where 0 stands for the measured lamination with empty support. It should be noted that $\mathcal{ML}(\partial M)$ contains measured laminations whose restriction to some component of ∂M is empty. The Teichmüller space $\mathcal{T}(\partial M)$ denotes similarly $\mathcal{T}(\partial_{\chi<0} M)$. The boundary of the Thurston compactification of $\mathcal{T}(\partial M)$ is equal to $\mathcal{PML}(\partial M)$.

2.4. **Some notations.** When λ is a measured geodesic lamination, we denote by $|\lambda|$ the support of λ. For an arc k whose intersections with $|\lambda|$ are transverse, we will denote by $\int_k d\lambda$ the λ-measure of k.

Let (u_n) and (v_n) be two sequences of non-negative real numbers. We say that u_n is $o(v_n)$ if for any ϵ there is $N(\epsilon)$ such that for $n \geq N(\epsilon)$, we have $u_n \leq \epsilon v_n$. We also write $u_n = o(v_n)$.

We will say that u_n is $O(v_n)$ if there are $K, N > 0$ such that for $n \geq N$, we have $u_n \leq K v_n$.

We say that u_n is $\Theta(v_n)$ if u_n is $O(v_n)$ and v_n is $O(u_n)$.

2.5. **Train tracks and their realisations.** A *train track* τ in a hyperbolic surface S is a union of finitely many rectangles with a distinguished pair of vertical opposite sides. These rectangles meet each other only along non-degenerate segments contained in their vertical sides in such a way that every point on a vertical side of a rectangle lies in at least one other rectangle. The rectangles are called *branches*, and they are foliated by vertical segments called *ties*. A connected component of the intersection of the branches is called a *switch*. The branches are also foliated by horizontal segments, and a smooth arc which is a union of horizontal segments is called a *rail* or a *train route*. A geodesic lamination is *carried* by τ if it is isotopic to one which lies in τ in such a way that the leaves are transverse to the ties (see [Bo] or [Ot3] for more details about train tracks).

When τ is a train track and λ is a measured geodesic lamination whose support is carried by τ, we say that λ is carried by τ. For a branch b of τ, we define the number $\lambda(b)$ to be the λ-measure of a tie of b. This number does not depend on the choice of a tie in b.

Consider an action of $\pi_1(S)$ on an $\mathbb{R}$-tree $\mathcal{T}$ by isometries. A measured lamination λ is said to be *realised* in $\mathcal{T}$ if there is a $\pi_1(S)$-equivariant map $\phi : \mathbb{H}^2 \to \mathcal{T}$ such that the restriction of ϕ to any lift of a leaf of λ in $\mathbb{H}^2$ is a complete geodesic in the tree, and identifying this geodesic with $\mathbb{R}$ gives a weakly monotonous unbounded function on the leaf. A train track $\tau \in S$ is said to be realised in $\mathcal{T}$ if there is an equivariant map $\phi : \mathbb{H}^2 \to \mathcal{T}$ which maps each lift of a branch of τ to a non-trivial geodesic segment on $\mathcal{T}$ in such a way that each rail is mapped injectively, and that each tie collapses to a point. By [Ot3], λ is realised in $\mathcal{T}$ if and only if λ is carried by a train track τ which is realised in $\mathcal{T}$.

We say that a measured lamination λ is *collapsed* by $\phi : \mathbb{H}^2 \to \mathcal{T}$ when there is a train track τ carrying λ such that every component of $\tilde{\tau}$, the preimage of τ in $\mathbb{H}^2$, is mapped to a point by ϕ. It is straightforward that there exists an equivariant map collapsing λ if and only if the action of $i_*(\pi_1(S(\lambda))$ on $\mathcal{T}$ has a global fixed point.

For a measured lamination μ on a surface S, if it does not have atoms, the semidistance on $\mathbb{H}^2$ induced by integrating the transverse measure $\tilde{\mu}$ along paths is continuous with respect to the usual topology of $\mathbb{H}^2$. By replacing closed leaves by annuli foliated by parallel closed curves, we obtain a measured partial foliation $\mathcal{F}_\mu$. The quotient of $\mathbb{H}^2$ under the semi-distance induced by $\mathcal{F}_\mu$ gives rise to a dual tree $\mathcal{T}_\mu$ with the projection $\pi : \mathbb{H}^2 \to \mathcal{T}_\mu$.

A morphism $\phi : \mathcal{T} \to \mathcal{T}'$ between $\mathbb{R}$-trees is a map with the property that every non-degenerate arc $[p, q] \subset \mathcal{T}$ contains a non-degenerate subarc $[p, r] \subset [p, q]$ which is mapped isometrically onto $\phi[p, r] \subset \mathcal{T}'$. A morphism from $\mathcal{T}_\mu$ to $\mathcal{T}$ is said to fold only at complementary regions if the only folding points are projections of complementary regions of $\tilde{\mathcal{F}}_\mu$, where p is a folding point if $[p, q], [p, q'] \subset \mathcal{T}_\mu, [p, q] \cap [p, q'] = \{p\}$ is mapped to the same segment in $\mathcal{T}$. The following theorems [MoO] will be useful to us later.

Theorem A [Morgan-Otal] *Let $(\alpha_1, \cdots, \alpha_k)$ be a collection of simple closed curves which define a pants decomposition of a surface S and let $\pi_1(S)$ act on an $\mathbb{R}$-tree $\mathcal{T}$. Then there is a measured lamination μ on S and an equivariant morphism*

$$\phi : \mathcal{T}_\mu \to \mathcal{T}$$

with $\delta_{\mathcal{T}}(\alpha_i) = \delta_{\mathcal{T}_\mu}(\alpha_i) = i(\mu, \alpha_i)$ for all i. Moreover ϕ folds only at complementary regions.

The above theorem is generalised by Skora [Sk].

Theorem B [Skora] *Suppose the action of $\pi_1(S)$ on an $\mathbb{R}$-tree $\mathcal{T}$ is minimal and small such that the action of each element representing ∂S has a fixed point in $\mathcal{T}$. Then there is a unique measured lamination μ and an equivariant isometry*

$$\phi : \mathcal{T}_\mu \to \mathcal{T}.$$

2.6. **Compactification of $\tilde{M}$.** We denote by $\tilde{M}$ the universal covering of M and by $p : \tilde{M} \to M$ the covering projection. We compactify $\tilde{M}$ in the following way: endow M with a geometrically finite hyperbolic metric σ with minimal parabolics which does not correspond to a Fuchsian representation, and let us denote by $N(\sigma)^{\text{thick}}$ the complement in the convex core $N(\sigma)$ of ϵ-thin neighbourhood of the cusps of σ for some ϵ smaller than the Margulis constant. Let us choose an isometry between the interior of $\tilde{M}$ and $\mathbb{H}^3$. Now we can consider $\tilde{N}(\sigma)^{\text{thick}}$ as a closed subset of $\mathbb{H}^3$. Since σ is geometrically finite, there is a natural homeomorphism between M and $N(\sigma)^{\text{thick}}$. Therefore we can regard $\tilde{M}$ as a closed subset of $\mathbb{H}^3$. The compactification

$\overline{\tilde{M}}$ of $\tilde{M}$ is the closure of this closed subset in the usual compactification of $\mathbb{H}^3$ by the unit ball. If we replace σ by another geometrically finite metric σ' with minimal parabolics, it follows from results of [Fl] that we get a compactification which is homeomorphic to the one obtained with σ. Therefore this definition is independent of the metric we chose. We call this the Floyd-Gromov compactification of $\tilde{M}$. To denote the ideal boundary in the Floyd-Gromov compactification, we use the symbol $\partial_\infty \tilde{M}$.

A *meridian* is a simple closed curve $c \subset \partial M$ which bounds a disc in M but not on ∂M. A compact surface $\Sigma \subset \partial M$ is incompressible if it contains no meridians. When we consider the closure of a lift of an incompressible surface in $\overline{\tilde{M}}$, we have the following:

Lemma 2.1. *Let $\Sigma \subset \partial_{\chi<0} M$ be a compact connected incompressible surface which does not contain any essential closed curve homotopic into $\partial_{\chi=0} M$. Let $\hat{\Sigma}$ be the universal covering of Σ, which is completed in the usual way to a closed disc $\overline{\hat{\Sigma}}$. Then any lift of Σ to $\partial \tilde{M}$ is a disc whose closure in $\overline{\tilde{M}}$ is homeomorphic to $\overline{\hat{\Sigma}}$ in an equivariant way.*

Proof. This lemma was proved in [Ot1]. See also Lemme 2.4 of [Le1]. □

Let $\Sigma \subset \partial M$ be a compact (possibly disconnected) incompressible surface. Johannson and Jaco-Shalen defined a characteristic submanifold W relative to Σ (cf. [Jo] and [JaS]). We say that an I-bundle $N \subset M$ is essential in (M, Σ) if

(1) $\pi_1(N)$ injects to $\pi_1(M)$ by the homomorphism induced from the inclusion,
(2) $N \cap \partial M \subset \Sigma$ is the associated ∂I-bundle, and
(3) N cannot be homotoped into Σ by a homotopy fixing $N \cap \Sigma$.

Similarly a solid torus $D^2 \times S^1$ is essential in (M, Σ) if it satisfies the first and the third conditions above.

When we are considering an atoroidal manifold M whose boundary contains no torus, a characteristic submanifold is a disjoint union of essential (possibly twisted) I-bundles over compact incompressible surfaces and essential solid tori. A disjoint union W of essential (possibly twisted) I-bundles and essential solid tori is said to be a characteristic submanifold if and only if it has the following two properties:

- any essential (possibly twisted) I-bundle and any essential solid torus in (M, Σ) can be homotoped in W;
- no connected component of W can be homotoped into another connected component of W.

By [Jo] and [JaS], if W and W' are two characteristic submanifolds relative to Σ, then there is a diffeomorphism $\psi : M \to M$ isotopic to the identity relative to $\partial M - \Sigma$ such that $\psi(W) = W'$ and that $\psi(W \cap \Sigma) = W' \cap \Sigma$.

Such a characteristic submanifold can be found by looking only at $\overline{\tilde{M}} - \tilde{M}$.

Proposition 2.2 ([Le1], §2, paragraphs after Lemme 2.7)**.** *Let Σ and $\Sigma' \subset \partial_{\chi<0}M$ be two compact, connected, incompressible surfaces which are disjoint or equal and do not contain any essential closed curve which can be homotoped into $\partial_{\chi=0}M$. Let $\tilde{\Sigma} \subset \partial\widetilde{M}$ (resp. $\tilde{\Sigma}'$) be a connected component of the preimage of Σ (resp. Σ') and let $\Gamma \subset \rho(\pi_1(M))$ (resp. Γ') be the stabiliser of $\tilde{\Sigma}$ (resp. Γ').*

Then $\overline{\tilde{\Sigma}} \cap \overline{\tilde{\Sigma}}'$ is either empty or equal to the limit set of $\Gamma \cap \Gamma'$.

In the latter case, if $\Gamma \cap \Gamma'$ is not cyclic, then it is the fundamental group of a (possibly twisted) I-bundle which is a connected component of a characteristic submanifold of $(M, \Sigma \cup \Sigma')$. If $\Gamma \cap \Gamma'$ is cyclic, then it is a finite index subgroup of a solid torus which is a connected component of a characteristic submanifold of $(M, \Sigma \cup \Sigma')$.

See Appendix for a brief proof of Proposition 2.2.

2.7. **Geodesic laminations on compressible surfaces.** Let M be a compact 3-manifold with boundary, and let $c \subset \partial M$ be a simple closed curve. A *c-wave* is a simple arc k, with $k \cap c = \partial k$ such that there is an arc κ in c with the simple closed curve $k \cup \kappa$ bounding a compressing disc in M. In some literature, a c-wave is allowed to intersect c in its interior. A simple innermost argument shows that if there is a c-wave in this generalised sense, there is one in our sense.

Let L be a geodesic lamination on $\partial_{\chi<0}M$, and let $c \subset \partial M$ be a multicurve. In the following, we always assume that simple closed curves or multicurves are geodesics for a fixed reference hyperbolic metric, hence there are no inessential intersections between them or with geodesic laminations. We say that L is in *tight position* with respect to c if L contains no c-waves and if *every* leaf of L intersects c transversely.

Following [Ot1, Theorem 1.6], we use cut-and-paste operations to construct a meridian m such that a given geodesic lamination contains no m-waves.

Claim 2.3. *Let $F \subset \partial M$ be a compressible compact surface, and let $\beta \subset F$ be a measured geodesic lamination. Then either β intersects transversely a meridian m and contains no m-waves, or there is a sequence of meridians $(m_i \subset F)$ converging in the Hausdorff topology to a geodesic lamination which does not cross β.*

Proof. If β intersects no meridians transversely, then, since F is compressible, there is a meridian $m \in F$ such that $i(\beta, m) = 0$. Setting $m_i = m$ for every i, we get the conclusion.

Now we assume that β intersects a meridian m transversely. If β contains an m-wave k, let us "cut m along k" to get a new meridian m_1 : let κ be the closure of a connected component of $m - \partial k$ such that $\int_\kappa d\beta \leq \frac{1}{2} i(\beta, m)$, and m_1 the simple closed geodesic in the free homotopy class of $k \cup \kappa$. We have $i(\beta, m_1) \leq \frac{1}{2} i(\beta, m)$ and m_1 is a meridian. If β contains no m_1-waves, we are done. If β contains an m_1-wave k_1, then we cut m_1 along k_1 as above to get a new meridian m_2 with $i(m_2,\beta) \leq \frac{1}{4} i(m,\beta)$. Repeating this operation, we get either a meridian m' such that β contains no m'-waves or a sequence of meridians (m_i) such that $i(m_i,\beta) \longrightarrow 0$. In the latter case, we can extract a subsequence such that m_i converges in the Hausdorff topology to some geodesic lamination H. Since $i(m_i,\beta) \longrightarrow 0$, we see that H does not cross β. □

Much of the rest of this section parallels the argument in [Le1], and we refer the readers there for more details.

We shall now introduce homoclinic leaves. They have already proved to be a useful tool in the study of laminations on the compressible boundary of a hyperbolic 3-manifold (see for example [Ot1], [KlS] and [Le1]). They are also related to Culler-Morgan-Shalen compactification of Character Varieties of freely decomposable Kleinian groups (see [KlS] and [Le1]). They shall play an important role in different places in this paper.

Let l be a simple geodesic on $\partial_{\chi<0} M$. Such a simple geodesic l is said to be *homoclinic* if a lift $\tilde{l}$ of l to the universal cover $\tilde{M}$ of M contains two sequences of point $(\tilde{x}_n)$ and $(\tilde{y}_n)$ such that the distance between $\tilde{x}_n$ and $\tilde{y}_n$ in $\tilde{M}$ is uniformly bounded whereas their distance measured on $\tilde{l}$ tends to ∞. By Lemma 2.1, an incompressible surface cannot contain a homoclinic geodesic.

Homoclinic leaves appear naturally in Hausdorff limits of sequences of meridians. This is illustrated in the following criterion of Casson whose proof can be found in [Ot1] and [Le1, Theorem B.1].

Lemma 2.4. *Let $(m_n \subset \partial M)$ be a sequence of meridians which converges to a geodesic lamination H in the Hausdorff topology. Then H contains a homoclinic leaf.*

A *simple half-geodesic* is an embedded half-line in ∂M whose image is locally geodesic for some hyperbolic metric on $\partial_{\chi<0} M$. Let $\tilde{l}_+ \subset \partial\tilde{M}$ be a half-geodesic and let $\bar{\tilde{l}}_+$ be its closure in the Floyd-Gromov compactification of $\tilde{M}$. We say that $\tilde{l}_+$ *has a well-defined endpoint* if $\bar{\tilde{l}}_+ - \tilde{l}_+$ contains one point. We say that a geodesic $\tilde{l} \subset \partial\tilde{M}$ has two well-defined endpoints if $\tilde{l}$ contains two disjoint half geodesics each having a well-defined endpoint. Notice that we allow the two endpoints to be the same.

As in the introduction, we say that a measured geodesic lamination $\lambda \in \mathcal{ML}(\partial M)$ is *doubly incompressible* if and only if there exists $\eta > 0$ such that $i(\lambda, \partial E) > \eta$ for every essential annulus, Möbius band or disc E in M.

We denote by $\mathcal{D}(M) \subset \mathcal{ML}(\partial M)$ the set of doubly incompressible measured geodesic laminations and by $\mathcal{PD}(M)$ its projection in the space $\mathcal{PML}(\partial M)$ of projective measured laminations.

Some properties of this set $\mathcal{D}(M)$ are discussed in [Le2]. One can deduce the following from [Le1].

Lemma 2.5. *Let $\lambda \in \mathcal{D}(M)$ be a measured geodesic lamination and $l_+, l_- \subset \partial M$ two simple half-geodesics which do not intersect $|\lambda|$ transversely. Then any lift of l_+ (resp. l_-) to $\tilde{M}$ has a well-defined endpoint in $\partial_\infty \tilde{M}$.*

Furthermore, if a lift of l_+ has the same endpoint as a lift of l_-, then they are asymptotic on $\partial\tilde{M}$.

Proof. The relation between doubly incompressible laminations and bending laminations is explained in [Le2, Lemma 3.5]. Knowing this relation, the proof of Lemma 2.5 can be deduced from [Le1] as follows.

The first property, namely that any lift of l_+ (resp. l_-) to $\tilde{M}$ has a well-defined endpoint in $\partial_\infty \tilde{M}$, can be deduced from the proofs of [Le1, Lemme 3.1] and [Le1, Lemme 3.3].

The proof of the second property, namely that if a lift of l_+ has the same endpoint as a lift of l_- then they are asymptotic on $\partial\tilde{M}$, can be found in Lemme C5 and more specifically in the paragraph after [Le1, Affirmation C3]. □

From [Le2], we also get the following:

Lemma 2.6. *Let $\lambda \in \mathcal{D}(M)$ be a measured geodesic lamination and h a homoclinic simple geodesic. Then the support $|\lambda|$ of λ crosses h.*

Proof. When M is not a genus-2 handlebody, this is [Le2, Lemma 3.6]. The case when M is a genus-2 handlebody is discussed in [Le2] in the remark following [Le2, Lemma 3.6]. □

By the same argument as the proof of Claim 2.3, we get the following.

Lemma 2.7. *Let λ be a measured geodesic lamination in $\mathcal{D}(M)$, and let $S \subset \partial M$ be a compressible surface. Then there is a meridian m in S such that S contains no m-waves disjoint from $|\lambda|$.*

Proof. Take any meridian m in S. If there is an m-wave in S which is disjoint from $|\lambda|$, then by the same argument as the proof of Claim 2.3, we get a sequence of meridians (m_i) on S with $i(m_i, \lambda) \longrightarrow 0$. This contradicts the assumption that λ is doubly incompressible. □

3. Realisations of doubly incompressible laminations

Let $\pi_1(M) \curvearrowright \mathcal{T}$ be a small minimal action of $\pi_1(M)$ on an $\mathbb{R}$-tree for a compact irreducible atoroidal 3-manifold M. Let S be a connected component of $\partial_{\chi<0} M$. Using the map $i_* : \pi_1(S) \to \pi_1(M)$ induced by the inclusion, we get an action of $\pi_1(S)$ on $\mathcal{T}$. Therefore, if $\lambda \in \mathcal{ML}(S)$ is a measured geodesic lamination, it makes sense to ask whether or not it is realised in $\mathcal{T}$. In this section, we shall discuss this question for the connected components of a measured lamination lying in $\mathcal{D}(M)$.

If a lamination is collapsed to a point by an equivariant map to a tree, then it is not realisable. Although the converse is not true, one may expect that non-realisability of a minimal component implies that the component is carried by a train track of bounded complexity whose edges are mapped to arbitrarily short arcs in the $\mathbb{R}$-tree. In the following lemma, we shall show that this expectation is fulfilled for a doubly incompressible lamination. The proof will occupy the rest of the section.

Lemma 3.1. *Let $\pi_1(M) \curvearrowright \mathcal{T}$ be a small minimal action of $\pi_1(M)$ on an $\mathbb{R}$-tree. Let $\lambda \in \mathcal{D}(M)$ be a measured geodesic lamination, and α a minimal sublamination of λ. Then one of the following holds:*

- *the measured lamination α is realised in $\mathcal{T}$;*
- *there is a sequence of train tracks θ_i each of which minimally carries α and has the following properties:*
  - *θ_i has only one switch κ_i and $\kappa_i \supset \kappa_{i+1}$ for every i.*
  - *There are a sequence $\eta_i \longrightarrow 0$, and a sequence of $\pi_1(S)$-equivariant maps $\phi_i : \mathbb{H}^2 \to \mathcal{T}$ such that ϕ_i maps every branch of the preimage of θ_i to a geodesic segment (which may be a point) with length smaller than η_i.*
  - *$\phi_i(\kappa_i) = x$ does not depend on i.*

Furthermore there is at least one component of λ which is realised in $\mathcal{T}$.

Note that, since θ_i has only one switch, the number of its branches is uniformly bounded.

Proof. When α is collapsed in $\mathcal{T}$, it is easy to see that we are in the second situation. The difficult case here is when α is neither collapsed nor realised in $\mathcal{T}$.

When α is a simple closed curve, either α is collapsed (when $\delta_{\mathcal{T}}(\alpha) = 0$) or α is realised (when $\delta_{\mathcal{T}}(\alpha) > 0$).

From now on, we assume that α is not a simple closed curve. As we will see later, the proof is easier when ∂M is incompressible. Bearing that in mind, we first get rid of the meridians that are disjoint from $S(\lambda)$. We cut M along a maximal family of compressing discs disjoint from $S(\alpha)$. We denote by N the connected component of the resulting manifold that contains α on its boundary. By construction, the surface $\partial N - S(\alpha)$ is incompressible in N.

Using the homomorphism $i_* : \pi_1(N) \to \pi_1(M)$ induced by the inclusion, we view $\pi_1(N)$ as a subgroup of $\pi_1(M)$. Thus we get a small action of $\pi_1(N)$ on $\mathcal{T}$. Let $\mathcal{T}_N$ be the minimal subtree of $\mathcal{T}$ that is invariant under the action of $\pi_1(N)$ regarded as a subgroup of $\pi_1(M)$.

We denote by F the component of ∂N that contains α. Since λ lies in $\mathcal{D}(M)$, no component of $\partial S(\alpha)$ bounds a disc in M. It follows that the measured lamination α can be regarded as an element of $\mathcal{ML}(F)$. Using the map $i_* : F \to N$ induced by the inclusion, we get an action of $\pi_1(F)$ on $\mathcal{T}_N$ (which is not small when ∂N is compressible). Let $\mathcal{T}_F$ be the minimal subtree of $\mathcal{T}_N$ that is invariant under the action of $\pi_1(F)$. If $\mathcal{T}_F$ is trivial, then α is collapsed. From now on we assume that $\mathcal{T}_F$ is not trivial.

As is often the case for 3-manifolds, the case when ∂N is incompressible is easier and will give us important insights for the general case. Since we are only interested in α, we just need to assume that F is incompressible.

First case : F is incompressible in N.

In this case, the action of $\pi_1(F)$ on $\mathcal{T}_F$ is small. By Theorem B (Skora's Theorem), there are a measured geodesic lamination β on F and an isomorphism $\phi : \mathcal{T}_\beta \to \mathcal{T}_F$ from the dual tree $\mathcal{T}_\beta$ of β to $\mathcal{T}_F$. If β crosses α, then α is realised in $\mathcal{T}$ (cf. [Ot3, Theorem 3.1.4]). If β and α are disjoint, then α is collapsed in $\mathcal{T}$.

It remains to deal with the case when α is a connected component of β. Let (η_i) be a sequence of positive numbers converging to 0. Let $\kappa_i \subset F$ be a sequence of segments intersecting α and β such that $\kappa_{i+1} \subset \kappa_i$ and $\int_{\kappa_i} d\beta \leq \frac{1}{2}\eta_i$. Let θ_i be a train track minimally carrying α and having only one switch, which is κ_i (refer to [BoO, § 3.2] for the construction of such a train track). Let p' be a point of $\bigcap_i \kappa_i$, and $\hat{p}' \in \mathbb{H}^2$ a lift of p'. This point $\hat{p}'$ corresponds to a point of $\mathcal{T}_\beta$ which we shall also denote by $\hat{p}'$. Let $p \in \mathcal{T}$ be the image of $\hat{p}'$ under ϕ. Let $\hat{\kappa}_i \subset \mathbb{H}^2$ be the lift of κ_i that contains $\hat{p}'$, and $\hat{\theta}_i$ the lift of θ_i that contains $\hat{\kappa}_i$. Define ϕ_i on $\hat{\kappa}_i$ by $\phi_i(\hat{\kappa}_i) = p$. Extend ϕ_i to an equivariant map from the union of the switches of $\hat{\theta}_i$ to $\mathcal{T}$. If $\hat{b}$ is a branch of $\hat{\theta}_i$, we define $\phi_i(\hat{b})$ to be the segment of $\mathcal{T}$ which connects the images of the vertical sides of $\hat{b}$. Finally, extend ϕ_i to a $\pi_1(S)$-equivariant map $\phi_i : \mathbb{H}^2 \to \mathcal{T}$.

We have thus constructed θ_i and ϕ_i, and we need to check that they have the expected properties. Let $\hat{b}$ be a branch of $\hat{\theta}_i$. Translating it by an element of $\pi_1(S)$, we can assume that $\hat{\kappa}_i$ contains a vertical side of $\hat{b}$. Since θ_i has only one switch, there is some $g \in \pi_1(S)$ such that $g(\hat{\kappa}_i)$ contains the other vertical side of $\hat{b}$. Let $\hat{k}_1$ be an arc joining $\hat{\kappa}_i$ to $g(\hat{\kappa}_i)$ whose projection k_1 on S lies in $b - |\beta|$. Then we have $\int_{k_1} d\beta = 0$. Let $\hat{k}_2 \subset \hat{\kappa}_i$ be an arc joining $\hat{p}'$ to $\hat{k}_1$ and let $\hat{k}_3 \subset g(\hat{\kappa}_i)$ be an arc joining $g(\hat{p}')$ to $\hat{k}_1$. The β-measures of k_2 and k_3 are less than $\int_{\kappa_i} d\beta \leq \frac{1}{2}\eta_i$. Therefore we have $\int_{k_1 \cup k_2 \cup k_3} d\beta \leq \eta_i$. This implies that the distance between $\hat{p}'$ and $g(\hat{p}')$ in $\mathcal{T}_\beta$ is less than η_i. It follows then from the construction of ϕ_i that the length of $\phi_i(\hat{b})$ is less than η_i.

Thus we conclude that when F is incompressible, one of the conclusions of Lemma 3.1 holds.

Second case : F is compressible.

The difference from the previous case is that the action of $\pi_1(F)$ on $\mathcal{T}_F$ is not small anymore, and hence we cannot use Theorem B directly. We shall show that if α is neither collapsed nor realised, the following conditions hold: $S(\alpha)$ is incompressible and each component d of $\partial S(\alpha)$ fixes a point in $\mathcal{T}$ (meaning that each element in the conjugacy class in $\pi_1(N)$ associated to

d has a fixed point). These conditions allow us to use Theorem B on $S(\alpha)$ and to construct θ_i as in the first case.

First we shall use Theorem A to associate a lamination to the action of $\pi_1(F)$ on $\mathcal{T}$.

Let (l_n) be a sequence of simple closed curves in $S(\alpha)$ converging to the support of α in the Hausdorff topology. By Theorem A (Morgan-Otal's Theorem), there exist, for any n, a measured lamination $\beta_n \in \mathcal{ML}(S(\alpha))$ and a morphism $\Phi_n : \mathcal{T}_{\beta_n} \to \mathcal{T}_N$, where $\mathcal{T}_{\beta_n}$ is the $\mathbb{R}$-tree dual to β_n, such that $\delta_{\mathcal{T}_{\beta_n}}(l_n) = \delta_{\mathcal{T}_N}(l_n)$. The difference from the case when F is incompressible is that Φ_n may not be an isomorphism and that β_n depends on l_n. Let B be the Hausdorff limit of $|\beta_n|$ after passing to a converging subsequence. If α is disjoint from B, then α is collapsed in $\mathcal{T}_N$. If α intersects B transversely, then α is realised in $\mathcal{T}$ as was shown in [KlS, Lemma 11]. Hence we only need to deal with the case when $|\alpha|$ is contained in B.

As will be explained below, it follows from results of [KlS], that B can be extended with a homoclinic leaf h. Combining this with the assumption that α is a sublamination of a doubly incompressible lamination we shall prove that $S(\alpha)$ is incompressible. Then we will use h to construct an essential I-bundle $W \subset N$ whose boundary contains $S(\alpha)$. Using W in the construction of β_n will give us enough control on β_n to guarantee that $i(\beta_n, \partial S(\alpha)) = 0$. This will imply that each component of $\partial S(\alpha)$ fixes a point in $\mathcal{T}$

Claim 3.2. *There is a geodesic lamination $H \supset B$ that contains a homoclinic leaf h.*

Proof. By Claim 2.3, either β_n is in tight position with respect to some meridian in F or there is a homoclinic geodesic $h_n \subset F$ which does not cross β_n. By the proof of [KlS, Proposition 2], if β_n intersects a meridian m and contains no m-waves, then $|\beta_n|$ can be extended to a geodesic lamination with a homoclinic leaf h_n. Furthermore by the proof of [KlS, Proposition 1], there is a Hausdorff limit of meridians that does not cross β_n. (See Appendix, Proposition 8.2 and Lemma 8.3.) Thus we have found in every case a sequence of meridians in F converging in the Hausdorff topology to a geodesic lamination H which does not have a transverse intersection with B. By Casson's criterion (Lemma 2.4), the lamination H contains a homoclinic leaf h. □

From now on we assume that $|\alpha|$ is a sublamination of B, since we have seen earlier that in the other cases, the conclusion is straightforward. Since α is a sublamination of a doubly incompressible lamination, h can not be a leaf of α (Lemma 2.6). As we saw in Lemma 2.1, an incompressible surface cannot contain a homoclinic geodesic. By construction $F - S(\alpha)$ is incompressible, hence the geodesic h does not lie in $F - S(\alpha)$. Since α is arational in $S(\alpha)$, this implies that there is a half-leaf h_+ of h which is asymptotic to a half-leaf of α on ∂N. Up to cutting h_+, we may assume that $h_+ \subset S(\alpha)$. Let h_- be another half-leaf of h which is disjoint from h_+ and from $\partial S(\alpha)$. From the assumption that λ is doubly incompressible, we shall deduce that h_- is disjoint from $S(\alpha)$.

Claim 3.3. *If $|\alpha| \subset B$, then $h_- \subset F - S(\alpha)$.*

Proof. Seeking a contradiction, we assume that $h_- \subset S(\alpha)$. We shall show that $F - S(\alpha)$ is compressible, which is impossible by construction.

We fix some geometrically finite hyperbolic metric on N and consider the Floyd-Gromov compactification of $\tilde{N}$. Let $\tilde{h}$ be a lift of h to $\tilde{N}$. Since h is homoclinic, the two endpoints of $\tilde{h}$ in $\partial_\infty \tilde{N}$ coincide. Let $\tilde{h}_\pm$ be the lift of $h_\pm$ that lies in $\tilde{h}$.

Since α is a sublamination of λ which is doubly incompressible, then by Lemma 2.5, $\tilde{h}_+$ and $\tilde{h}_-$ are asymptotic on $\tilde{F}$. Take a short geodesic arc $\tilde{k}$ connecting $\tilde{h}_-$ and $\tilde{h}_+$ which does not lie on $\tilde{h}$ so that they form a triangle on $\partial\tilde{N}$ with one vertex at infinity in $\partial_\infty\tilde{N}$. Then any lift of a half-leaf of α entering this triangle must have the same endpoint in $\partial_\infty\tilde{N}$ as the lifts of $\tilde{h}_-, \tilde{h}_+$ since h does not intersect α transversely. Therefore such half-leaf of α is trapped between $\tilde{h}_-$ and $\tilde{h}_+$. It follows that we can push $\tilde{k}$ towards the end of $\tilde{h}_\pm$ without changing the intersection number with $\tilde{\alpha}$, where $\tilde{\alpha}$ is the preimage of α. Thus we obtain a sequence of geodesic arcs $(\tilde{k}_i)$, whose lengths tend to zero and whose α-measure is $\int_{\tilde{k}_i} d\tilde{\alpha} = \int_{\tilde{k}} d\tilde{\alpha}$. We project it to ∂N and pass to a subsequence so that (k_i) converges to a point in the Hausdorff topology. Since the transverse measure of α is non-atomic, the only way this can happen is that $\int_{\tilde{k}} d\tilde{\alpha} = 0$, i.e., $\tilde{\alpha}$ lies outside the triangle and k is disjoint from α.

Let $\tilde{k}'$ be the segment on $\tilde{h}$ between the two endpoints of $\tilde{k}$ lying on $\tilde{h}$. Then $\tilde{k}$ is homotopic to $\tilde{k}'$ in $\tilde{N}$ since $\tilde{N}$ is simply connected. Let k, k' be the projections of $\tilde{k}, \tilde{k}'$ to ∂N, and m be the closed geodesic homotopic to $k \cup k'$. We see that k is homotopic to k' in N but not on ∂N since both of them are geodesic arcs. Therefore, m bounds a (possibly singular) disc in N, and it is disjoint from α since both h and k are disjoint from α. This contradicts the fact that $F - S(\alpha)$ is incompressible by the Loop Theorem. □

It is quite easy to deduce from this Claim that $S(\alpha)$ is incompressible.

Claim 3.4. *If $|\alpha| \subset B$, then $S(\alpha)$ is incompressible.*

Proof. Let us assume the contrary. By Lemma 2.7, there is a meridian $m \subset S(\alpha)$ such that $S(\alpha)$ contains no m-waves disjoint from α. In particular h_+ is in tight position with respect to m. Let $(\tilde{m}_n) \subset \tilde{N}$, $n \in \mathbb{N}$, be the family of lifts of m which intersect $\tilde{h}_+$. Since h_+ is in tight position with respect to m, each $\tilde{m}_n$ intersects $\tilde{h}_+$ once, the curves $\tilde{m}_n$ are nested and converge to the endpoint of $\tilde{h}_+$ ([Le1, Claim 3.4]). Since $\tilde{h}_+$ and $\tilde{h}_-$ have the same endpoint, $\tilde{h}_-$ intersects $\tilde{m}_n$ for n large enough. In particular h_- intersects $m \subset S(\alpha)$, contradicting Claim 3.3. □

We denote by A the minimal lamination contained in the closure of h_-. By Claim 3.3, $S(A)$ is disjoint from $S(\alpha)$. Since $F - S(\alpha)$ is incompressible, so is $S(A)$. We shall use Proposition 2.2, to construct an essential I-bundle whose boundary contains $S(A)$ and $S(\alpha)$.

Claim 3.5. *If $|\alpha| \subset B$, then there is an essential I-bundle $W \subset N$ whose associated ∂I-bundle is $S(\alpha) \cup S(A)$. Moreover, W is homeomorphic to $T \times I$ with T homeomorphic to $S(\alpha)$*

Proof. Let $\widetilde{S(\alpha)}, \widetilde{S(A)}$ be lifts of $S(\alpha), S(A)$ containing $\tilde{h}_+, \tilde{h}_-$ respectively. Since $S(\alpha)$ and $S(A)$ are incompressible, by Lemma 2.1, the closures $\overline{\widetilde{S(\alpha)}}, \overline{\widetilde{S(A)}}$ of $\widetilde{S(\alpha)}, \widetilde{S(A)}$ in $\overline{\tilde{N}}$ are discs.

Since $\tilde{h}_+$ and $\tilde{h}_-$ have the same endpoint in $\partial_\infty \tilde{N}$, the two discs $\overline{\widetilde{S(\alpha)}}$ and $\overline{\widetilde{S(A)}}$ intersect in $\partial_\infty \tilde{N}$. By Proposition 2.2, there is an essential I-bundle W embedded in $(N, S(\alpha) \cup S(A))$, and h_+ and h_- are contained in $W \cap (S(\alpha) \cup S(A))$ (after cutting off arcs of finite lengths from h^+ and h^-). Since α and A lie in the closures of h_+ and h_- respectively, the corresponding ∂I-bundle contains $S(\alpha) \cup S(A)$. Hence $W \cap \partial N = S(\alpha) \cup S(A)$. This can happen only when W is homeomorphic to $T \times I$ with T homeomorphic to $S(\alpha)$. $\square$

Since $|\alpha|$ lies in the closure of h_+ and A lies in the closure of h_-, for any leaf $\tilde{a}$ of the preimage of α in $\widetilde{S(\alpha)}$, there is a leaf $\tilde{a}'$ of the preimage of A with the same endpoints. It follows that we can isotope the bundle structure so that the projections of A and the support of α along the fibres of W coincide.

Now we can prove that each component of $\partial S(\alpha)$ has a fixed point in $\mathcal{T}_N$.

Claim 3.6. *If $|\alpha| \subset B$, then $\delta_{\mathcal{T}_N}(\partial S(\alpha)) = 0$.*

Proof. We choose a simple closed curve $c \subset S(A)$ which is not homotopic to a component of $\partial S(A)$. In the construction of β_n, we add the conditions that $\delta_{\mathcal{T}_{\beta_n}}(c) = \delta_{\mathcal{T}_N}(c)$ and that $\delta_{\mathcal{T}_{\beta_n}}(d) = \delta_{\mathcal{T}_N}(d)$ for any component d of $\partial(T \times \{0\} \cup T \times \{1\})$. Notice that this has no effect on the proof of Claim 3.2 and all that follows (in particular Claim 3.5).

If β_n does not intersect $\partial S(A)$ for some n, then $\delta_{\mathcal{T}_{\beta_n}}(\partial S(A)) = 0 = \delta_{\mathcal{T}_N}(\partial S(A))$. By Claim 3.5, $\delta_{\mathcal{T}_N}(\partial S(\alpha)) = \delta_{\mathcal{T}_N}(\partial S(A)) = 0$, and we are done.

Otherwise, B intersects $S(A)$. Since h does not cross B and is asymptotic to A, this implies that A lies in B. In particular, any leaf of B intersecting $\partial S(A)$ contains a half-leaf asymptotic to A. Such a half-leaf intersects the simple closed curve $c \subset S(A)$ infinitely many times. This implies that $i(\beta_n, \partial S(A))$ is $o(i(\beta_n, c))$. On the other hand, by construction, we have $i(\beta_n, c) = \delta_{\mathcal{T}}(c)$ for any n. Thus we get $i(\beta_n, \partial S(A)) \longrightarrow 0$. However, by assumption, $i(\beta_n, \partial S(A)) = \delta_{\mathcal{T}_N}(\partial S(A))$ does not depend on n. Thus we have $\delta_{\mathcal{T}_N}(\partial S(A)) = 0 = \delta_{\mathcal{T}_N}(\partial S(\alpha))$. $\square$

It follows that the conjugacy class in $\pi_1(F)$ represented by each component of $\partial S(\alpha)$ has a fixed point in $\mathcal{T}_N$. This enables us to use Theorem B on the minimal subtree of $\mathcal{T}_N$ which is invariant under the action of $\pi_1(S(\alpha))$. Then we can use the same arguments as in the First case to construct the train tracks θ_i and maps ϕ_i and show that the second alternative in the statement of Lemma 3.1 holds.

Thus we have proved the first part of Lemma 3.1. It only remains to show that at least one component of λ is realised in $\mathcal{T}$. This was already proved in [Le2, Proposition 6.1]. Let us briefly review the proof. Let (L_n) be a sequence of multi-curves converging to λ in the Hausdorff topology. As we have already seen, by Theorem A there exist a measured geodesic lamination $\beta_n \in \mathcal{ML}(\partial M)$ and a morphism $\phi_n : \mathcal{T}_{\beta_n} \to \mathcal{T}$ from the dual tree of β_n to $\mathcal{T}$ such that for any simple closed curve l_n which lies in L_n, either $\delta_{\mathcal{T}}(l_n) > 0$ and the restriction of ϕ_n to the axis of l_n is an isometry or $\delta_{\mathcal{T}}(l_n) = 0$ and $i(l_n, \beta_n) = 0$. Extract a subsequence such that (β_n) converges in the Hausdorff topology to a geodesic lamination B. As we have seen above, any connected component of λ that intersects B transversely is realised in $\mathcal{T}$.

If $S(B)$ is compressible, then, by the proof of [KlS, Proposition 2], $S(B)$ contains a homoclinic geodesic h which does not cross B. Such a homoclinic leaf crosses λ by Lemma 2.6. Thus, if $S(B)$ is compressible, λ crosses B.

If $S(B)$ is incompressible, then we can apply Skora's Theorem B to each component of $S(B)$. It follows that β_n does not depend on n for sufficiently large n. Denote by β this constant geodesic measured lamination β_n. We deduce then from [MoS2] that β is an annular lamination (see [BoO, Démonstration du Lemme 14]). Thus we see that λ crosses the support B of β in this case as well. □

4. Mapping the train tracks to $\epsilon_n\mathbb{H}^3$

In this section, we shall begin the proof of Theorem 2. Here, we consider a situation more general than the setting in Theorem 2 as we shall explain below. We consider a sequence of measured laminations (λ_n) on S converging to λ. We also assume that $|\lambda_n|$ converges to a geodesic lamination L_∞ in the Hausdorff topology and that each minimal sublamination of L_∞ satisfies the conclusion of Lemma 3.1. Let L_{rec} be the union of the recurrent leaves of L_∞.

We shall use the train tracks obtained in lemma 3.1 to construct a sequence of train tracks τ_n carrying λ and L_∞ which are decomposed into three parts. The first part τ^1 is independent of n and carries the realised part of L_{rec}. The second part τ_n^2 carries the non-realised part of L_{rec}. The third part τ_n^3, which is $\tau_n - (\tau^1 \cup \tau_n^2)$ and not a train track, carries the rest of L_∞, and has weights with smaller order. Let us denote by $\hat{\tau}_n$ the preimage of τ_n under the covering projection from the universal cover of ∂M to ∂M.

In the process of constructing τ_n, we shall also build ρ_n-equivariant maps $\tilde{f}_n$ from $\hat{\tau}_n$ to $\mathbb{H}^3$ which map the branches of $\hat{\tau}_n^1$ to long segments and the branches of $\hat{\tau}_n^2$ to much shorter ones. These properties make it possible to estimate the lengths of measured laminations carried by these train tracks.

Recall that when $\hat{\tau} \subset \tilde{S}$ is the preimage of a train track τ on a component S of ∂M, we say that a map $\hat{h} : \hat{\tau} \rightarrow \mathbb{H}^3$ is ρ_n-equivariant if and only if for every $g \in \pi_1(S)$ and $x \in \hat{\tau}$, we have $\hat{h}(gx) = \rho_n(i_*(g))\hat{h}(x)$, where i denotes the inclusion from S to M.

We shall work in the following setting: (ρ_n) is a sequence of geometrically finite representations uniformising M. There is an ultrafilter ω and $\epsilon_n \longrightarrow 0$ such that the action of ρ_n on $\epsilon_n\mathbb{H}^3$ tends to a small minimal action of $\pi_1(M)$ on an $\mathbb{R}$-tree $\mathcal{T}$ with respect to ω. We have a sequence (λ_n) of measured geodesic laminations converging to a measured lamination λ. Furthermore $|\lambda_n|$ converges in the Hausdorff topology to a geodesic lamination L_∞ and each minimal sublamination of L_∞ satisfies the conclusion of Lemma 3.1.

We call these assumptions the *light assumptions*. We use the assumption that each minimal sublamination of L_∞ satisfies the conclusion of Lemma 3.1 rather than the one that λ is doubly incompressible because it will not contradict the assumption that $l_{\rho_n}(\lambda_n^*)$ is bounded which we shall add in the next section. This way the statements of lemmas and claims in Sections 5 and 6 is non-empty, i.e. not based on contradictory hypothesis. To illustrate this statement, let us construct an example with these light assumptions which has bounded $l_{\rho_n}(\lambda_n^*)$.

Suppose that M is a handlebody which we regard as an I-bundle $T \times I$ over a compact surface with boundary T. We pick a convex cocompact representation ρ uniformising M, a meridian m, and two pseudo-Anosov diffeomorphisms $\varphi, \psi : T \to T$. We extend φ and ψ to fibred diffeomorphisms $\varphi_M, \psi_M : M \to M$. We set $\rho_n = \psi_{M*}^n \circ \rho$. This sequence tends to the action of $\pi_1(M) \approx \pi_1(T)$ on the $\mathbb{R}$-tree dual to the stable lamination of ψ. We set $m_n = \varphi_M^n(m)$, pick a simple closed curve $c \subset \partial M$ which is not entirely contained in $T \times \{0\}$ nor in $T \times \{1\}$, and consider the sequence $\lambda_n = D_{m_n}^{p_n} \circ \psi_M^n(c)$ where $D_{m_n} : \partial M \to \partial M$ is the right Dehn twist along m_n. If p_n is large enough, compared to n, the Hausdorff limit L_∞ of λ_n contains two minimal sublaminations both of which are projected along the fibres to the stable lamination of φ. Each of such sublaminations satisfies the conclusion of Lemma 3.1 (it is easy to extend each of them to a doubly incompressible lamination abandoning the other). On the other hand λ_n is homotopic to $\psi_M^n(c)$, hence $l_{\rho_n}(\lambda_n^*) = l_\rho(c^*)$ is bounded.

We detail the properties of τ_n and $\tilde{f}_n$ in the following lemma. We shall work under the light assumptions which were introduced earlier. The rest of the section is devoted to the proof of Lemma 4.1, which consists in constructing τ_n and $\tilde{f}_n$.

Lemma 4.1. *Under the light assumptions, after taking a subsequence of* (ρ_n)*, there are train tracks* τ_n *carrying both* L_∞ *and* λ_n *with decomposition* $\tau_n = \tau^1 \cup \tau_n^2 \cup \tau_n^3$ *such that*

- *each minimal sublamination of* L_∞ *is carried by* τ^1 *or* τ_n^2*;*
- λ_n *is minimally carried by* τ_n*;*
- τ^1 *and* τ_n^2 *are disjoint subtracks of* τ_n *(we do not require* τ_n^3 *to be a subtrack);*
- τ_n^3 *is the union of all branches of* τ_n *that are not contained in* $\tau^1 \cup \tau_n^2$*;*
- *the switches of* τ_n *lie in* $\tau^1 \cup \tau_n^2$*;*
- *the sum of the weights with which* τ_n *carries* λ_n *is bounded independently of* n*;*

and there is a ρ_n*-equivariant map* $\tilde{f}_n : \hat{\tau}_n \to \mathbb{H}^3$ *from the preimage of* τ_n *in the universal cover of* ∂M *to* $\mathbb{H}^3$*, which has decomposition* $\hat{\tau}_n = \hat{\tau}^1 \cup \hat{\tau}_n^2 \cup \hat{\tau}_n^3$ *corresponding to the decomposition* $\tau_n = \tau^1 \cup \tau_n^2 \cup \tau_n^3$*, such that:*

a) *for any branch* $\hat{b}$ *of* $\hat{\tau}_n$*, its image* $\tilde{f}_n(\hat{b})$ *is either a geodesic segment or a point;*

b) *there are* $R > 0$ *and* $n(R) \in \mathbb{N}$ *such that, for* $n \geq n(R)$*, if* $\hat{b}$ *is a branch of* $\hat{\tau}^1$*, then* $l(\tilde{f}_n(\hat{b})) \geq R\epsilon_n^{-1}$*, where* ϵ_n *is the rescaling factor that appeared in the light assumptions;*

c) *there is a sequence of positive numbers* $\delta_n \longrightarrow 0$ *such that, for any* $n \in \mathbb{N}$*, if* $\hat{b}_1, \hat{b}_2$ *are two adjacent branches of* $\hat{\tau}^1$ *which are separated by a switch, then the exterior angle between* $\tilde{f}_n(\hat{b}_1)$ *and* $\tilde{f}_n(\hat{b}_2)$ *is less than* δ_n*;*

d) *there is a sequence of positive numbers* $\eta_n \longrightarrow 0$ *such that, for any* $n \in \mathbb{N}$*, if* $\hat{b}$ *is a branch of* $\hat{\tau}_n^2$*, then we have* $\epsilon_n l(\tilde{f}_n(b)) \leq \eta_n$*;*

e) *for any* $n \in \mathbb{N}$*, if* $\hat{b}$ *is a branch of* $\hat{\tau}_n^3$*, then* $\lambda_n(b)(\epsilon_n l(\tilde{f}_n(\hat{b})))$ *is less than* η_n *for* (η_n) *given in (d).*

Proof. It was proved by Thurston that the set of weighted simple closed curve is dense in $\mathcal{ML}(S)$ (cf. [FLP] and [Pe]), for every component S of ∂M. By approximating each (λ_n) by a sequence of such unions of weighted simple closed curves, and by a diagonal extraction, we get a sequence of unions of weighted simple closed curves satisfying the light assumption. Therefore we can assume that for each component S of ∂M, the intersection $\lambda_n \cap S$ is either empty or a weighted simple closed curve.

Since we have only to construct train tracks on each component of ∂M with non-empty intersection with L_∞, we can assume that L_∞ is contained in a component S of ∂M. Let L be a minimal sublamination of L_{rec}.

Let us first consider the case when there is a train track θ minimally carrying L which is realised in $\mathcal{T}_S$ (recall that $\mathcal{T}_S$ is the minimal subtree of $\mathcal{T}$ invariant under $i_*\pi_1(S)$). Let $\hat{\theta}$ be a lift of θ to the universal cover $\mathbb{H}^2$ of S. There is a continuous $\pi_1(S)$-equivariant map $\phi_n : \mathbb{H}^2 \to \mathcal{T}_S$ under ρ_n such that ϕ_n is constant on every tie of $\hat{\theta}$ and the restriction of ϕ_n to any rail is injective. Following [Ot2], we shall construct a ρ_n-equivariant map $\tilde{f}_n : \hat{\theta} \to \mathbb{H}^3$. Let $\kappa_1, ..., \kappa_p$ be the switches of θ and $\hat{\kappa}_1, ..., \hat{\kappa}_p \subset \mathbb{H}^2$ lifts of $\kappa_1, ..., \kappa_p$. Denote by $\tilde{x}_{i,n} \in \mathcal{T}_S \subset (X_\omega, x)$ the point $\phi_n(\hat{\kappa}_i)$. We first define $\tilde{f}_n$ on $\{\hat{\kappa}_1, ..., \hat{\kappa}_p\}$ by setting $\tilde{f}_n(\hat{\kappa}_i) = \tilde{x}_{i,n}$. We extend this map to $\pi_1(S)(\{\hat{\kappa}_1, ..., \hat{\kappa}_p\})$ by $\tilde{f}_n(g(\kappa_i)) = \rho_n(g) \circ \tilde{f}_n(\hat{\kappa}_i)$ for any $g \in \pi_1(S)$ and any $1 \le i \le p$. Let $\hat{b}$ be a branch of $\hat{\theta}$. The vertical sides of $\hat{b}$ lie in two switches $\hat{\kappa}$ and $\hat{\kappa}'$ whose images by $\tilde{f}_n$ have already been defined. On $\hat{b}$, we let $\tilde{f}_n$ be the map which is constant on each tie of $\hat{b}$, and which induces a parametrisation of the geodesic segment joining $\tilde{f}_n(\hat{\kappa})$ to $\tilde{f}_n(\hat{\kappa}')$ with constant speed on a horizontal side of $\hat{b}$. Then for any branch $\hat{b}$ of $\hat{\theta}$, we have

$$\epsilon_n l(\tilde{f}_n(\hat{b})) \longrightarrow l_{\mathcal{T}_S}(\phi_n(\hat{b})) > 0.$$

Let θ' be the first subdivision of θ as defined in [Ot3, Chapitre 4, §4.1]: that is, for each branch b_j of θ, starting from every endpoint of $b_j \cap b_k$ lying on the vertical side of b_j, we cut b_j along a horizontal arc up to its midpoint and reorganise the decomposition into branches keeping the condition that it carries λ_n. We let $\hat{\theta}' \subset \hat{\theta}$ a lift of θ'. We shall deform the map $\tilde{f}_n$ to one which is adapted to $\hat{\theta}'$. For a branch $\hat{b}$ of $\hat{\theta}'$, its image by $\tilde{f}_n$ is a broken geodesic segment which is the union of two geodesic segments. We deform $\tilde{f}_n$ by a homotopy which is constant on the vertical sides of $\hat{b}$ to a map which is constant on each tie of $\hat{b}$ and takes $\hat{b}$ to the geodesic segment joining the images under $\tilde{f}_n$ of the vertical sides of $\hat{b}$. By slightly abusing notation, we shall denote the deformed map by the same symbol $\tilde{f}_n$.

Since θ is realised in $\mathcal{T}_S$, it follows from the argument of [Ot3, Chapitre 4] that $\tilde{f}_n$ has the following properties:

b) there are $R > 0$ and $n(R)$ such that, for $n \ge n(R)$, if $\hat{b}$ is a branch of $\hat{\theta}'$, we have $l(\tilde{f}_n(\hat{b})) \ge R\epsilon_n^{-1}$;
c) there is a sequence of positive numbers $\delta_n \longrightarrow 0$ such that if $\hat{b}_1, \hat{b}_2$ are two adjacent branches of $\hat{\theta}$ which are separated by a switch, then the external angle between $\tilde{f}_n(\hat{b}_1)$ and $\tilde{f}_n(\hat{b}_2)$ is smaller than δ_n.

We repeat the same construction for all the components of L_{rec} that are realised in $\mathcal{T}_S$. Denote by τ^1 the union of the train tracks θ' thus obtained, by $\hat{\tau}^1$ its preimage in $\mathbb{H}^2$, and by $\tilde{f}_n : \hat{\tau}^1 \to \mathbb{H}^3$ the map which agrees with the map defined above on each connected component of $\hat{\tau}^1$. By Lemma 3.1, τ^1 is not empty. We also see that λ_n passes through every branch of τ^1 for every n after taking a subsequence.

When a component L of L_{rec} is not realised in $\mathcal{T}$, Lemma 3.1 gives rise to a sequence of train tracks θ_i each carrying L minimally. We can assume that λ_i passes through every branch of θ_i. Let us denote by τ_i^2 the union of the train tracks thus obtained from the components of L_{rec} which are not realised in $\mathcal{T}$. Finally we add branches to $\tau^1 \cup \tau_i^2$ to get a train track τ_i which minimally carries L_∞ and λ_i.

We shall now extend the map $\tilde{f}_n$ to the preimage $\hat{\tau}_i^2$ of τ_i^2. Consider a connected component L of L_{rec} which is not realised in $\mathcal{T}$. Consider the subtrack θ_i of τ_i^2 which minimally carries L. We get from Lemma 3.1 that there are a point $x \in \mathcal{T}$, a sequence $\eta_i \longrightarrow 0$, and a sequence of $\pi_1(S)$-equivariant maps $\phi_i : \mathbb{H}^2 \to \mathcal{T}$ such that ϕ_i maps each branch of the preimage of θ_i to a geodesic segment (which may be a point) with length smaller than η_i and a lift of κ_i (the switch of θ_i) is mapped to x under ϕ_i. Set $x = (\tilde{x}_n) \in \Pi_n(\epsilon_n \mathbb{H}^3)$ (we pick an element in the equivalence class defined by x) and fix some $i \in \mathbb{N}$. Let $\hat{\theta}_i \subset \mathbb{H}^2$ be a lift of θ_i, and $\hat{\kappa}_i \subset \hat{\theta}_i$ the lift of κ_i which is mapped to x by ϕ_i.

Let $G_i \subset \pi_1(\partial M)$ be a finite set consisting of all $g \in \pi_1(\partial M)$ such that if $\hat{b}$ is a branch of $\hat{\theta}_i$ and $\hat{\kappa}_i$ contains a vertical side of $\hat{b}$, then the other vertical side of $\hat{b}$ lies in $g(\hat{\kappa}_i)$. Recall that for each branch $\hat{b}$, the length of $\phi_i(\hat{b})$ is less than η_i. Therefore, we have $d(x, gx) \leq \eta_i$ for any $g \in G_i$. Since $\mathcal{T}$ is the ω-limit of $\epsilon_n \mathbb{H}^3$, we have $\epsilon_n d(\tilde{x}_n, \rho_n(g)(\tilde{x}_n)) \leq 2\eta_i$ for any $g \in G_i$ for n large enough. For any $g \in G_i \cup \{id\}$, we define $\tilde{f}_n(g(\hat{\kappa}_i))$ by $\tilde{f}_n(g(\hat{\kappa}_i)) = \rho_n(g)(\tilde{x}_n)$. Let $\hat{b}$ be a branch of $\hat{\theta}_i$ with vertical sides lying in two switches $\hat{\kappa}_i$ and $g(\hat{\kappa}_i)$ for some $g \in G_i$. If $\tilde{f}_n(\hat{\kappa}_i) = \tilde{f}_n(\rho_n(g)(\hat{\kappa}_i))$, then we set $\tilde{f}_n(\hat{b}) = \tilde{f}_n(\hat{\kappa}_i)$. Otherwise, we set $\tilde{f}_n$ to be the map which is constant on each tie of $\hat{b}$ and takes $\hat{b}$ to the geodesic segment joining $\tilde{f}_n(\hat{\kappa}_i)$ to $\tilde{f}_n(\rho_n(g)(\hat{\kappa}_i))$. Extend $\tilde{f}_n$ to an equivariant map from $\hat{\theta}_i$ to $\mathbb{H}^3$. For sufficiently large n and any branch $\hat{b}$ of $\hat{\theta}_i$, we have $\epsilon_n l(\tilde{f}_n(b)) \leq 2\eta_i$. Furthermore, by construction, the sum of the weights with which θ_i carries λ_n is bounded by $\int_{\kappa_i} d\lambda_n \leq \int_{\kappa_1} d\lambda_n \longrightarrow \int_{\kappa_1} d\lambda$.

We do the same construction for all the components of L_{rec} that are not realised in $\mathcal{T}_S$, and denote by $\tilde{f}_n : \hat{\tau}^1 \cup \hat{\tau}_i^2 \to \mathbb{H}^3$ the maps whose restriction to each connected component of $\hat{\tau}^1 \cup \hat{\tau}_i^2$ is the maps thus defined. It follows from the construction that there is $N(i)$ such that for $n \geq N(i)$, if $\tilde{b}$ is a branch of $\hat{\tau}_i^2$, we have $\epsilon_n l(\tilde{f}_n(\tilde{b})) \leq 2\eta_i$.

We set τ_i^3 to be the closure of $\tau_i - (\tau^1 \cup \tau_i^2)$, and $\hat{\tau}_i^3$ its preimage in $\mathbb{H}^2$. It remains to define $\tilde{f}_n$ on the branches of $\hat{\tau}_i^3$. Let $\hat{b}$ be a branch of $\hat{\tau}_i^3$. Let $\hat{\kappa}$ and $\hat{\kappa}'$ be the two vertical sides of $\hat{b}$. Their projections κ and κ' lie in $\tau^1 \cup \tau_i^2$. Hence their images by $\tilde{f}_n$ are already defined, and there are two points $x = (\tilde{x}_n), x' = (\tilde{x}'_n)$ in $\mathcal{T}$ such that $\tilde{f}_n(\bar{\kappa}) = \tilde{x}_n$ and $\tilde{f}_n(\bar{\kappa}') = \tilde{x}'_n$. We set $\tilde{f}_n$ to be the map which is constant on each tie of $\hat{b}$ and takes $\hat{b}$ to the geodesic segment joining $\tilde{x}_n$ to $\tilde{x}'_n$. We then have $\epsilon_n d(\tilde{x}_n, \tilde{x}'_n) \longrightarrow d(x, x')$. Furthermore, since λ is carried by $\tau^1 \cup \tau_i^2$, we have $\lambda_n(b) \longrightarrow 0$. Therefore, for n large enough, we have $\lambda_n(b)(\epsilon_n l(\tilde{f}_n(\tilde{b}))) \leq 2\eta_i$.

Thus we have proved that there is $N(i)$ such that for $n \geq N(i)$, for a branch b of $\tau_i - \tau^1$, either b is a branch of τ_i^2 and we have $\epsilon_n l(\tilde{f}_n(\hat{b})) \leq 2\eta_i$ or b is a branch of τ_i^3 and we have $\lambda_n(b)(\epsilon_n l(\tilde{f}_n(\hat{b}))) \leq 2\eta_i$. Now by choosing $N(i)$ such that $N(i) < N(i+1)$, and taking a subsequence $\lambda_{N(n)}$ so that the

n-th term is the original $N(n)$-th term, we obtain the desired train track. This concludes the proof of Lemma 4.1. $\square$

5. Finding backtracking

In this section, we are going to show that for large enough n, the path $f_n(c_n)$ has long segments in which it comes back nearly parallel to itself. Eventually these close returns will allow us to construct some long and thin strips connecting two segments of $f_n(c_n)$. Here we use the adjective "long" not only for the rescaled metric of $\epsilon_n\mathbb{H}^3$ but also to mean combinatorially long in the sense that they go through many branches of τ_n. Let us fix some notations and definitions to put this idea into a precise statement.

We consider the train tracks τ_n and maps $\tilde{f}_n$ which come from Lemma 4.1. Take a component S of ∂M such that $S \cap \tau^1 \neq \emptyset$. In this section, we only have to pay attention to the behaviour of λ_n on S. Therefore, for simplicity, we denote $\lambda_n \cap S$ by λ_n, and $\tau_n \cap S$ by τ_n, etc. Furthermore, λ_n is assumed to be a weighted simple closed curve with support c_n and weight w_n.

Let $f_n : \tau_n \longrightarrow \mathbb{H}^3/\rho_n(\pi_1(M))$ be the projection of $\tilde{f}_n$. By construction, τ_n carries c_n. We set $c_n^1 = c_n \cap \tau^1$, $c_n^2 = c_n \cap \tau_n^2$ and $c_n^3 = c_n \cap \tau_n^3$. We set $\bar{c}_n^j = f_n(c_n^j)$ for $j = 1, 2, 3$ and $\bar{c}_n = f_n(c_n) = \bar{c}_n^1 \cup \bar{c}_n^2 \cup \bar{c}_n^3$.

Fix an orientation on c_n. Let s be a segment lying in c_n^1 with the orientation induced by that on c_n. The *train route* $b(1), ..., b(t)$ *of* s is the ordered finite sequence of branches of τ^1 through which s passes: $b(i)$ is an element of the set $\{b_1, ..., b_p\}$ of branches of τ^1. We fix an orientation for each branch of τ^1. A branch $b(i)$ in the train route of s is said to be *positively oriented* if its orientation coincides with the orientation of s and *negatively oriented* otherwise. The *oriented train route* $bo(1), ..., bo(t)$ of s is the ordered finite sequence of oriented branches of τ^1 through which s goes in this order with the assigned orientations: $bo(i)$ is an element of $\{b_1, ..., b_p\} \times \{+, -\}$. When $(bo(i))_{i\in I}$ is an oriented train route, we shall denote by $(b(i))_{i\in I}$ the corresponding non-oriented train route.

In the following lemma, we shall show that $f_n(c_n)$ nearly backtracks along some long path. Using the terms of oriented train routes, this is expressed as follows.

Lemma 5.1. *Under the light assumptions, if* $(l_{\rho_n}(\lambda_n^*))$ *is bounded, then there are two infinite oriented train routes* $bo, bo' : \mathbb{N} \to \{b_1, ..., b_p\} \times \{+, -\}$ *in* τ^1 *and functions* $T, V : \mathbb{N} \to \mathbb{N}$ *such that for any* $n \in \mathbb{N}$ *there are two disjoint segments* $s_n, s'_n \subset c_n^1$ *satisfying the following:*

- T *is non-decreasing and unbounded;*
- *the oriented train routes of* s_n *and* s'_n *are* $(bo(i))_{0\leq i\leq T(n)}$ *and* $(bo'(i))_{0\leq i\leq V(T(n))}$ *respectively;*
- *there is a homeomorphism* $g_n : f_n(s_n) \to f_n(s'_n)$*;*
- $g_n(f_n(v(i))) \in f_n(b'(V(i)))$ *for any* $i \leq T(n)$ *where* $v(i) = b(i) \cap b(i+1)$*;*

- *any point* $x \in f_n(s_n)$ *is connected to* $g_n(x)$ *by an essential arc* $\zeta_n(x) \subset M_n$ *with length less than* 6ϵ*;*
- *the simple closed curve* $f_n(s_n) \cup \zeta_n(f_n(\partial s_n)) \cup f_n(s'_n)$ *bounds a disc* D_n *containing all the arcs* $\zeta_n(x)$ *for* $x \in f_n(s_n)$*.*

Recall that λ_n is assumed to be a weighted simple closed curve with support c_n and weight w_n. We denote by c_n^* the geodesic representative of c_n in $M_n = \mathbb{H}^3/\rho_n(\pi_1(M))$. Then we set $l_{\rho_n}(\lambda_n^*) = w_n l_{\rho_n}(c_n^*)$ by definition.

Notice that we have adopted the conventions that $0 \in \mathbb{N}$, and that if $T(n) = 0$, there are no segments s_n and s'_n.

We shall see that $V(i) \leq i\frac{R'}{R} + 1$ for any $i \in \mathbb{N}$ (equation (2)).

Proof. First we shall show that most points x in $\bar{c}_n^1$ are close to another component of $\bar{c}_n^1$ (i.e. not the component containing x). The proof goes roughly as follows: we construct a simplicial annulus A_n between $\bar{c}_n$ and c_n^*. If a point x of $\bar{c}_n^1$ is not close in A_n to a point in another component of $\bar{c}_n^1$ then either x is close to a component of $\bar{c}_n^2$ or $\bar{c}_n^3$ or x is not close to any component of $\bar{c}_n^j$ for $j = 1, 2, 3$. Using the Gauss-Bonnet formula and the length comparison between the components of $\bar{c}_n^j$, we shall show that this can happen only for the minority of the points of $\bar{c}_n^j$.

Let us start the formal proof. The curve $\bar{c}_n = f_n(c_n)$ is a piecewise geodesic. We define the edges of $\bar{c}_n$ to be the images of the intersections of c_n with the branches of τ_n, and the vertices to be the images of the intersections of c_n with the switches of τ_n.

Let $x_{n,1}, ..., x_{n,p_n}$ be the vertices of $\bar{c}_n$, and choose the same number of points $y_{n,1}, ..., y_{n,p_n}$ on c_n^*. We shall make a strip with boundaries $(x_{n,i})$ and $(y_{n,i})$ and triangulate it by making each rectangle into a pair of triangles. To be more precise, for $1 \leq i \leq p_n$, we consider the geodesic triangle with vertices $y_{n,i}, x_{n,i}, x_{n,i+1}$ (with $x_{n,p_n+1} = x_{n,1}$ and $y_{n,p_n+1} = y_{n,1}$) and the geodesic triangle with vertices $x_{n,i+1}, y_{n,i}, y_{n,i+1}$. The union of these triangles for $i = 1, \ldots, p_n$ is a simplicial annulus $A_n = S^1 \times [0, 1]$ bounded by c_n^* and $\bar{c}_n$. The metric ν_n induced on this annulus by the lengths of paths is a hyperbolic metric with piecewise geodesic boundary. By the Gauss-Bonnet formula, the area of A_n is less than $2p_n\pi$. By Lemma 4.1, the sequence $(w_n p_n) = (\sum_{b\text{: the branches of } \tau_n} \lambda_n(b))$ is bounded. We parametrise A_n by $S^1 \times [0, 1]$ so that the projection of $S^1 \times \{1\}$ to $\mathbb{H}^3/\rho_n(\pi_1(M))$ is c_n^*.

For a positive number ϵ, which we shall specify later, and each point $x \in \bar{c}_n^1$, we consider a geodesic arc a_x on $(S^1 \times I, \nu_n)$ perpendicular to $S^1 \times \{0\}$ at x having length ϵ with respect to ν_n. If the perpendicular reaches $S^1 \times \partial I$ before the length ϵ is attained, we define a_x to be the geodesic arc having both endpoints on $S^1 \times \partial I$.

We shall show that most arcs a_x issuing from a component of $\bar{c}_n^1$ intersect an arc a_y issuing from another component of $\bar{c}_n^1$. For this purpose, we are going to estimate from below the length of the set of points x for which the a_x reach $S^1 \times \{1\}$ without intersecting themselves or each other. Since the length of this set of points is bounded above by length(c_n^*) (with respect to ν_n), we get an inequality, which will appear as the inequality (i) below. For that, we need to subtract from the length of $\bar{c}_n^1$ the lengths of (I) the set of points x for which a_x has self-intersection, (II) the set of points x for which a_x intersects a_y with $x \neq y$, (III) the set of points x for which a_x has an endpoint on either $\bar{c}_n^2$ or $\bar{c}_n^3$, and (IV) the set of points x which are neither of type (I) nor of type (II) and for which a_x has an endpoint in the interior of $S^1 \times I$. See Figure 1.

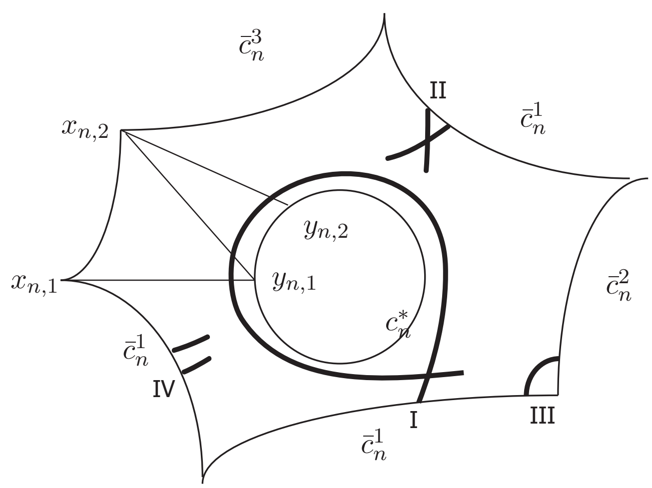

FIGURE 1. Annulus between c_n^* and $f_n(c_n)$

We first consider the contribution of the points of type (I) to the length, i.e., x for which a_x intersects itself transversely. By the Gauss-Bonnet formula, a geodesic loop formed by a subarc of a_x cannot be null-homotopic. Hence, there must be a loop formed by a subarc of a_x freely homotopic to $S^1 \times \{1\}$. It follows that if both of two perpendiculars a_{x_1}, a_{x_2} with $x_1 \neq x_2$ have self-intersection, then $a_{x_1} \cap a_{x_2} \neq \emptyset$. Thus, the contribution of the set of x with self-intersecting a_x (i.e. of type (I)) to the length is absorbed in the contribution of x with a_x intersecting another a_y, that is, of type (II), which will be dealt with below.

We next consider the points x of type (II), i.e. those for which a_x intersects a_y for some $y \in \bar{c}_n^1$. Let m be a point in the intersection $a_x \cap a_y$, and let a'_x, a'_y be subarcs of a_x, a_y between x and m and y and m respectively. Let β be an arc on $\bar{c}_n$ to which $a'_x \cup a'_y$ is homotopic fixing the endpoints. Suppose that β is also contained in $\bar{c}_n^1$. We then say that x is an *inessential* point of type (II) and that $a'_x \cup a'_y$ is an *inessential arc*. It was shown in [Bo, Lemme 5.11] that, in this situation, there is a constant ξ_n depending only on ϵ and the maximal exterior angle of the vertices on $\bar{c}_n^1$, which is less than δ_n in our case, such that x is within distance ξ_n with respect to ν_n from a vertex of $\bar{c}_n^1$. It was also shown in [Bo] that the constant ξ_n tends to 0 as either δ_n or ϵ goes to 0. If β does not lie on $\bar{c}_n^1$, then we say that x is an *essential* point of type (II). See Figure 2.

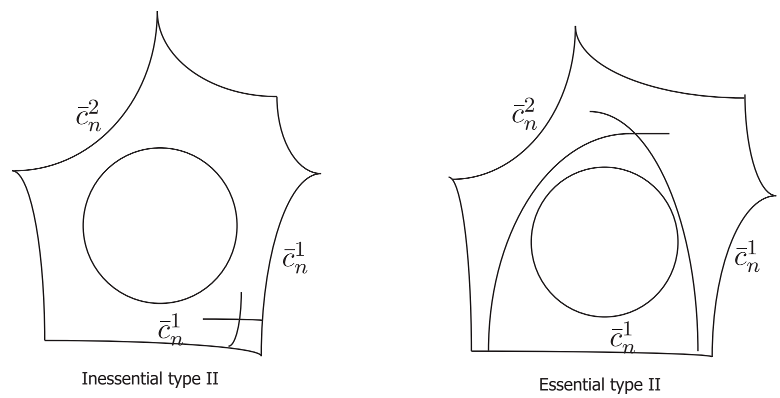

FIGURE 2. Inessential and essential points of type II.

We note that the length of an essential arc (such as $a'_x \cup a'_y$) is less than or equal to 2ϵ. Let $\bar{c}_n^+$ be the union of the essential points of type (II). We shall use the essential points of type (II) to construct the long and thin strips of Lemma 5.1.

Now we shall bound the length of the sets of points of type (III) and (IV). First we consider the points of type (III). The total length with respect to ν_n of the set of points x on $\bar{c}_n^1$ for which a_x reaches a point on $\bar{c}_n^2$ is bounded above by the length of $\bar{c}_n^2$. Similarly, the total length of the set of points x for which a_x reaches a point on $\bar{c}_n^3$ is bounded above by the length of $\bar{c}_n^3$.

Finally we consider the points of type (IV); the points x such that $a_x \setminus \{x\}$ is contained in the interior of $S^1 \times I$ while a_x has neither self-intersection nor intersection with another a_y. Since the union of a_x for x of type (IV) has area bounded below by the length of the set of points x of type (IV) multiplied by $\mathrm{sh}(\epsilon)$, we can bound the length from above by $\mathrm{Area}(A_n)/\mathrm{sh}(\epsilon)$.

Putting all of these considerations together, we get an inequality:

$$\text{(i)} \quad l_{\rho_n}(\bar{c}_n^1) - 2p_n\xi_n - l_{\rho_n}(\bar{c}_n^+) - l_{\rho_n}(\bar{c}_n^2) - l_{\rho_n}(\bar{c}_n^3) - \mathrm{Area}(A_n)/\mathrm{sh}(\epsilon) \leq l_{\rho_n}(c_n^*).$$

Notice that we have

$$w_n l_{\rho_n}(\bar{c}_n^2) = \sum_{b:\ \text{branches of } \tau_n^2} \lambda_n(b) l_{\rho_n}(f_n(b))$$

and
$$w_n l_{\rho_n}(\bar{c}_n^3) = \sum_{b\text{: branches of } \tau_n^3} \lambda_n(b) l_{\rho_n}(f_n(b)).$$
Therefore, by the property (d) of Lemma 4.1, we have $w_n l_{\rho_n}(\bar{c}_n^2) = o(\epsilon_n^{-1})$ and by the property (e), we have $w_n l_{\rho_n}(\bar{c}_n^3) = o(\epsilon_n^{-1})$.

By Lemma 4.1 $(w_n p_n)$ is a bounded sequence. It follows that we have $2w_n p_n \xi_n \longrightarrow 0$. This implies also that $w_n \mathrm{Area}(A_n) \leq 2w_n p_n \pi$ is a bounded sequence and that we have $\epsilon_n w_n \mathrm{Area}(A_n) \longrightarrow 0$.

By assumption $(w_n l_{\rho_n}(c_n^*)) = (l_n(\lambda_n))$ is a bounded sequence; hence $\epsilon_n w_n l_{\rho_n}(c_n^*)$ tends to 0. Thus we have shown the following.

Claim 5.2. *We have* $w_n \epsilon_n (l_{\nu_n}(\bar{c}_n^1) - l_{\nu_n}(\bar{c}_n^+)) \longrightarrow 0.$ □

Now that we know that $\bar{c}_n^+$ occupies the most part of $\bar{c}_n^1$, we shall use $\bar{c}_n^+$ to construct maps g_n and strips D_n. First we define a discrete version of g_n. Let $\{\sigma_{1,n}, \sigma_{2,n}, \dots\} \subset \bar{c}_n$ be a maximal family of disjoint segments with diameter 6ϵ such that the midpoint $x_{i,n}$ of $\sigma_{i,n}$ lies in $\bar{c}_n^+$. Consider a segment $\sigma_{i,n}$ and its middle point $x_{i,n}$. In the family of essential arcs joining $x_{i,n}$ to $\bar{c}_n^1$, we take an arc $a_{i,n}$ to be the shortest (with respect to ν_n). Since $x_{i,n}$ lies in $\bar{c}_n^+$, the length of $a_{i,n}$ is less than 2ϵ.

If $\partial a_{i,n} - x_{i,n}$ lies in some $\sigma_{j,n}$, we denote by $\zeta_{i,n}$ the geodesic segment in $(S^1 \times I, \nu_i)$ joining $x_{i,n}$ to $x_{j,n}$ which is homotopic to $a_{i,n}$ relative to $x_{i,n} \cup \sigma_{j,n}$. If $\partial a_{i,n} - x_{i,n}$ is disjoint from $\bigcup \sigma_{j,n}$, we define $\zeta_{i,n}$ to be $a_{i,n}$ (see Figure 3). The length of each segment $\zeta_{i,n}$ thus obtained is less than 5ϵ. Using the minimality of the lengths of the arcs $a_{i,n}$, we shall next show that the segments $\zeta_{i,n}$ have mutually disjoint interiors.

Consider two different segments $\zeta_{1,n}$ and $\zeta_{2,n}$ and assume that their interiors intersect. Then the interiors of $a_{1,n}$ and $a_{2,n}$ also intersect. Let y be a point in the intersection. Let $[x_{\ell,n}, y[$, $\ell = 1, 2$ be the connected component of $a_{\ell,n} - \{y\}$ containing $x_{\ell,n}$. Let γ_n be the shortest of the two segments $a_{1,n} - [x_{1,n}, y[$ and $a_{2,n} - [x_{2,n}, y[$ (see Figure 3). Then, for $\ell = 1, 2$, the length of the arc $[x_{\ell,n}, y] \cup \gamma_n$ is less than or equal to the length of $a_{\ell,n}$. Furthermore one of the two arcs $[x_{\ell,n}, y] \cup \gamma_n$, say $[x_{1,n}, y] \cup \gamma_n$ is not the shortest in its homotopy class relative to the endpoints. Let a_1' be the shortest arc homotopic to $[x_{1,n}, y] \cup \gamma_n$ relative to the endpoints. Then the length of the segment a_1' (with respect to ν_n) is less than the length of $a_{1,n}$. Recall that we chose $a_{1,n}$ which is shortest among all essential arcs joining $x_{1,n}$ to $\bar{c}_n^1$. It follows that a_1' is not essential, i.e. there is a segment $\beta \subset \bar{c}_n^1$ homotopic to a_1' relative to the endpoints. The endpoints of β are $x_{1,n}$ and another point which we call y_1. The distance (with respect to ν_n) between $x_{1,n}$ and y_1 is less than the length of a_1' which is less than 2ϵ. By the properties (b) and (c), each component of $\bar{c}_n^1$ (in particular the one containing β) is a union of long geodesic segments such that the exterior angle between two consecutive segments is small. By [CEG, Lemma 4.2.10] such a component of $\bar{c}_n^1$ is a (K, η)-quasi-geodesic with $K \to 1, \eta \to 0$ as $n \to \infty$. It follows that there is N

(independent of β) such that for $n \geq N$, the length of β is less than 3ϵ. This implies that y_1 lies in $\sigma_{1,n}$. By our definition of $\zeta_{1,n}$, it has an endpoint on $x_{1,n}$, not on y_1 (see Figure 3). This contradicts our assumption that the interiors of $\zeta_{1,n}$ and $\zeta_{2,n}$ intersect.

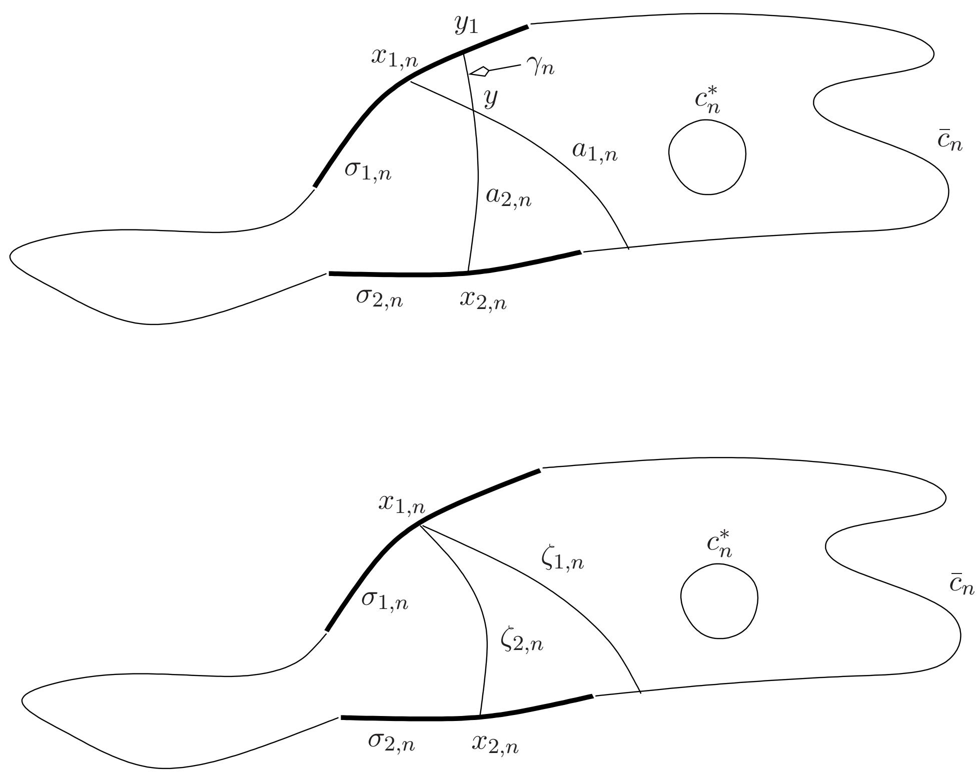

FIGURE 3. From $a_{i,n}$ to $\zeta_{i,n}$

Even if some segment $a_{i,n}$ has a self-intersection, the same argument shows that $\zeta_{i,n}$ does not have any self-intersection.

Thus we have proved the following claim.

Claim 5.3. *There are a map* $h_n : \{x_{1,n}, x_{2,n}, ...\} \to \bar{c}_n^1$ *and a family* $(\zeta_{i,n})$ *of essential segments with disjoint interiors such that the length of* $\zeta_{i,n}$ *is less than* 5ϵ*,* $\partial\zeta_{i,n} = \{x_{i,n}, h_n(x_{i,n})\}$*, and* $\{h_n(x_{1,n}), h_n(x_{2,n}), ...\} \cap \bigcup_i \sigma_{i,n} \subset \{x_{1,n}, x_{2,n}, ...\}$. □

We shall use those segments $\zeta_{i,n}$ as a skeleton to construct our strips D_n. To this end we need to fill the space between two consecutive segments $\zeta_{i,n}$ and $\zeta_{i+1,n}$ with a thin strip. In particular, we need to check that $\zeta_{i,n}$ and $\zeta_{i+1,n}$ are in the same homotopy class and that each point of $\bar{c}_n^1$ between $x_{i,n}$ and $x_{i+1,n}$ is close to a point on $\bar{c}_n$ between $h_n(x_{i,n})$ and $h_n(x_{i+1,n})$. Our proof will rely mainly on counting segments. Namely we start with sufficiently many segments and prove that at each step,there are only few among them which fail to satisfy the property that we need.

Let j be a positive integer, and cut $\bar{c}_n^1$ into disjoint segments $\bar{s}_{1,n}, \bar{s}_{2,n}, \ldots$, each containing j edges (if the number of edges of some component of $\bar{c}_n^1$ is not a multiple of j, then there are some edges of $\bar{c}_n^1$ not belonging to any one of these segments).

We shall evaluate the number of segments thus obtained using the following claim.

Claim 5.4. *Let r_n be the number of the components of $\bar{c}_n^1$. Then, $w_n r_n \longrightarrow 0$ as $n \to \infty$.*

Proof. Note that any train route on τ_n connecting a point in τ^1 and a point in τ_n^2 must pass through a point in τ_n^3. Therefore between any two distinct components of c_n^1, there is a component of c_n^3. Hence $(w_n r_n)$ is bounded above by $\sum_{b\subset\tau_n^3} \lambda_n(b)$. Since $|\lambda| \subset L_{rec}$ is carried by $\tau^1 \cup \tau_n^2$, the sum $\sum_{b\subset\tau_n^3} \lambda_n(b)$ tends to 0 as $n \to \infty$. It follows that we have $w_n r_n \longrightarrow 0$. □

Since

$$w_n(\text{number of edges of } \bar{c}_n^1) = w_n(\sum_{b\subset\tau^1} c_n(b)) \longrightarrow \sum_{b\subset\tau^1} \lambda(b), \tag{1}$$

the number of edges of $\bar{c}_n^1$ is $\Theta(w_n^{-1}) = \Theta(p_n)$ (see Section 2.4 for notations), where p_n was defined to be the number vertices on $\bar{c}_n$ (i.e. the number of times c_n crosses a switch of τ_n). The number of edges of $\bar{c}_n^1$ lying in none of the $\bar{s}_{i,n}$ is less than $jr_n = o(p_n)$. It follows that the number of the segments $\bar{s}_{i,n}$ is $\Theta(p_n)$.

Let t_n be the number of edges of $\bar{c}_n^1$ containing no one among the segments $\sigma_{i,n}$ defined earlier. If an edge e contains no segment among the $\sigma_{i,n}$, then there is no point of $\bar{c}_n^+$ in e outside the 3ϵ-neighbourhood of ∂e. By the property (b) in Lemma 4.1, the total length of these edges is greater than $t_n R\epsilon_n^{-1}$ and smaller than $l_{\nu_n}(\bar{c}_n^1) - l_{\nu_n}(\bar{c}_n^+) + 6t_n\epsilon$. By Claim 5.2 and the equation (1), we have $w_n\epsilon_n(l_{\nu_n}(\bar{c}_n^1) - l_{\nu_n}(\bar{c}_n^+) + 6t_n\epsilon) \longrightarrow 0$. Hence $w_n\epsilon_n t_n R\epsilon_n^{-1} \longrightarrow 0$ and t_n is $o(w_n^{-1}) = o(p_n)$.

Thus we know that among the $\bar{s}_{i,n}$, there are $\Theta(p_n)$ disjoint segments lying in $\bar{c}_n^1$ and containing j edges (where j is the number we have fixed when cutting $\bar{c}_n^1$) each of which contains a segment among the $\sigma_{i,n}$. We shall denote these $\Theta(p_n)$-many segments again by $\bar{s}_{1,n}, \bar{s}_{2,n}, \ldots$.

Next we shall show that most of these segments are joined to only one component of $\bar{c}_n^1$ through the arcs $\zeta_{i,n}$.

Let $\bar{s} \in \{\bar{s}_{1,n}, \bar{s}_{2,n}, \ldots\}$ be a segment with the following property: there are at least two distinct components of $\bar{c}_n^1$ containing an endpoint of $\zeta_{i,n}$ for some $x_{i,n} \in \bar{s}$. Let t'_n be the number of those with this property among the $\bar{s}_{i,n}$. In each such segment $\bar{s}$, we choose two points in $\{x_{1,n}, x_{2,n}, \ldots\} \cap \bar{s}$, say $x_{1,n}, x_{2,n}$, such that $\zeta_{1,n}$ and $\zeta_{2,n}$ connect $\bar{s}$ to distinct components of $\bar{c}_n^1$

and the segment $]x_{1,n}, x_{2,n}[\subset \bar{s}$ contains no $x_{k,n}$. We note that there may be $\zeta_{i,n}$ other than $\zeta_{1,n}, \zeta_{2,n}$ which have $x_{1,n}$ or $x_{2,n}$ as an endpoint. There are two points $y_{1,n}$ and $y_{2,n}$ which lie in two distinct components of $\bar{c}_n^1$ such that $y_{1,n}$ (resp. $y_{2,n}$) is connected to $x_{1,n}$ or $x_{2,n}$ by some $\zeta_{i,n}$ and that $y_{1,n}$ and $y_{2,n}$ are innermost in the following sense : if $[y_{1,n}, y_{2,n}]$ is the segment of $\bar{c}_n$ joining $y_{1,n}$ to $y_{2,n}$ whose interior does not contain $x_{1,n}$, then there is no $\zeta_{i,n}$ connecting $]y_{1,n}, y_{2,n}[$ to $\{x_{1,n}, x_{2,n}\}$ (these $y_{1,n}, y_{2,n}$ may or may not coincide with $h_n(x_{1,n}), h_n(x_{2,n})$). Let us embed A_n into a round disc in such a way that $\bar{c}_n$ is the boundary of the disc, and connect $y_{1,n}$ to $y_{2,n}$ by a geodesic segment with respect to the ordinary Euclidean metric on the disc.

Assume that for another segment $\bar{s}' \in \{\bar{s}_{1,n}, \dots\}$ with the same property, the resulting geodesic segment in the round disc intersects transversely the geodesic segment produced from $\bar{s}$ above (i.e. the geodesic segment connecting $y_{1,n}$ to $y_{2,n}$). Suppose that $x_{3,n}$ and $x_{4,n}$ are the points on $\bar{s}'$ chosen in the same way as $x_{1,n}, x_{2,n}$ for $\bar{s}$. We number them so that the order in which the four points lie on the circle is $x_{1,n}, x_{2,n}, x_{3,n}, x_{4,n}$. Since $\zeta_{i,n}$ have disjoint interiors and $y_{1,n}$ and $y_{2,n}$ are innermost, we see that the only ways this can happen are the following two: (1) $x_{3,n} = y_{1,n}$ and (2) $x_{4,n} = y_{2,n}$. Therefore for each $\bar{s}$, there are only two configurations of $\bar{s}'$ such that the geodesics in the round disc intersect transversely. See Figure 4.

Thus we can build at least $\frac{1}{3}t'_n$ disjoint geodesic segments in the round disc, each connecting two distinct components of $\bar{c}_n^1$. Furthermore, since the $\zeta_{i,n}$ have disjoint interiors, any pair of connected components of $\bar{c}_n^1$ is connected by at most two of these disjoint segments. Consider a map from $\bar{c}_n$ to a round circle which preserves the order and maps each connected component C_i of $\bar{c}_n^1$ to a point Q_i. Join two points Q_i and Q_j by a segment if and only if there is one of the segments constructed above which joins C_i and C_j. Thus we have constructed at least $\frac{1}{6}t'_n$ segments with disjoint interiors. We can now add some geodesic segments in the disc to get a triangulation of the polygon with vertices $Q_1, ..., Q_{r_n}$. Such a triangulation has $2r_n - 3$ edges (this can easily be computed with the Euler formula). Therefore we have $\frac{1}{6}t'_n \leq 2r_n - 3 = o(p_n)$.

Since we initially had $\Theta(p_n)$ segments $\{\bar{s}_{1,n}, \bar{s}_{2,n}, ...\}$, after excluding $o(p_n)$ segments as above from them, there remains, by abusing notations again, $\Theta(p_n)$ disjoint segments $\{\bar{s}_{1,n}, \bar{s}_{2,n}, ...\}$ in $\bar{c}_n^1$, such that if $\bar{s}$ is one of those segments, then we have:

- $\bar{s}$ contains j edges;
- i) each edge of $\bar{s}$ contains some $x_{i,n}$;
- ii) there is a unique component C (depending on $\bar{s}$) of $\bar{c}_n^1$ such that for any $x_{i,n} \in \bar{s}$, we have $h_n(x_{i,n}) \in C$.

Let $\bar{s}$ be one of these segments, C the associated component of $\bar{c}_n^1$, and x a point of $\bar{s} \cap \{x_{1,n}, ...\}$. Denote by ζ_x the corresponding segment $\zeta_{i,n}$. Since A_n is an annulus and ζ_x is embedded, there are only two possibilities for the homotopy class of ζ_x relative to $\bar{s} \cup C$. Therefore, taking $2j$ instead of

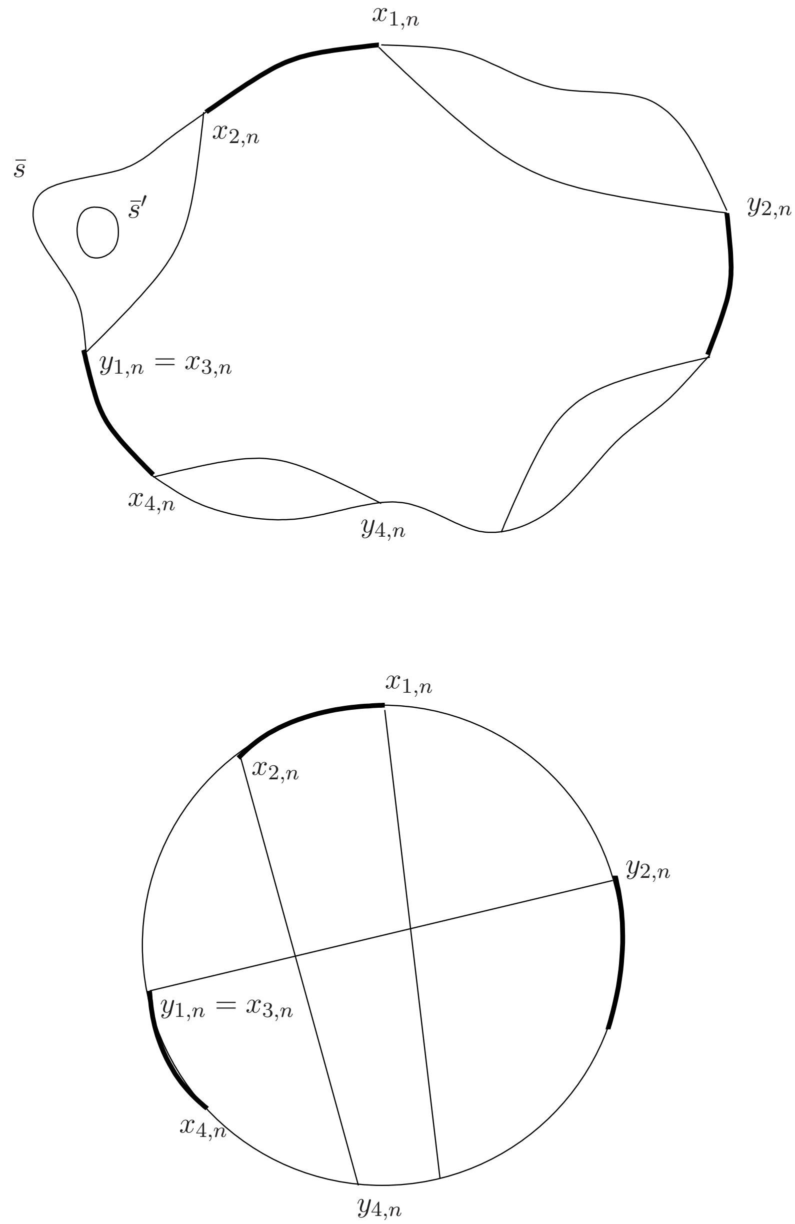

FIGURE 4. A segment intersects at most two other segments

j at the beginning and cutting each segment into two groups, we get $\Theta(p_n)$ disjoint segments $\{\bar{s}_{1,n}, ...\}$ in $\bar{c}_n^1$, each one containing j edges and satisfying (i), (ii) above and:

iii) for any $x, y \in \bar{s} \cap \bar{c}_n^+$, ζ_x and ζ_y are homotopic relative to $\bar{s} \cup C$.

Now we have a sufficiently number of j consecutive segments $\sigma_{i,n}$ with the properties (i), (ii) and (iii). Next we shall show that they lie in the boundary of the expected strips D_n.

Let $\bar{s}$ be one of the segments produced above, and C the corresponding component of $\bar{c}_n^1$ in the property (ii). Let x and y be the extremal points of $\bar{s} \cap \{x_{1,n}, ...\}$, and $[x, y]$ the segment in $\bar{s}$ joining x to y. The segment $[x, y]$ contains at least $(j-2)$ edges. We have the following:

Lemma 5.5. *There is $N \in \mathbb{N}$ which does not depend on $\bar{s}$ such that for any $n \geq N$ we have the following by homotoping A_n keeping ∂A_n and $\bar{c}_n^i$ unchanged:*
There is a homeomorphism $g_n : [x, y] \to [h_n(x), h_n(y)]$ such that for any $n \geq N$ and any $z \in [x, y]$, the two points z and $g_n(z)$ are connected by an essential arc whose length (with respect to the induced metric ν_n on A_n) is less than 6ϵ.

Proof. By the property (iii), the simple closed curve $\zeta_x \cup [x, y] \cup \zeta_y \cup [h_n(x), h_n(y)]$ bounds a disc in A_n. Since both $[x, y]$ and $[h_n(x), h_n(y)]$ lie in $\bar{c}_n^1$, by the properties (b) and (c) in Lemma 4.1, they consist of long geodesic segments such that the external angles formed by two adjacent segments are less than δ_n. Let k be the geodesic segment in $\mathbb{H}^3/\rho_n(\pi_1(M))$ joining x to y which is homotopic to $[x, y]$, and let k' be the one in $\mathbb{H}^3/\rho_n(\pi_1(M))$ joining $h_n(x)$ to $h_n(y)$ which is homotopic to $[h_n(x), h_n(y)]$. We parametrise the arcs $[x, y]$, $[h_n(x), h_n(y)]$, k and k' by their arc lengths.

By [CEG, Lemma 4.2.10], for sufficiently large n, we have $d([x, y](t), k(t)) \leq \epsilon'_n$ and $d([h_n(x), h_n(y)](t), k'(t)) \leq \epsilon'_n$ for any t with $\epsilon'_n \longrightarrow 0$. It follows that $1 \leq \dfrac{l([x, y])}{l(k)} \leq \dfrac{l(k) + \epsilon'_n}{l(k)} \leq 1 + \epsilon'_n$ and that $1 \leq \dfrac{l([h_n(x), h_n(y)])}{l(k')} \leq 1 + \epsilon'_n$ for sufficiently large n, where $l(.)$ denotes the length in $\mathbb{H}^3/\rho_n(\pi_1(M))$. Therefore, we have $d([x, y](\dfrac{l([x, y])}{l(k)} t), k(t)) \leq 2\epsilon'_n$ for sufficiently large n. For the same reason, we have also

$$d([h_n(x), h_n(y)](\frac{l([h_n(x), h_n(y)])}{l(k')} t), k'(t)) \leq 2\epsilon'_n.$$

By the property (iii), the simple closed curve $\zeta_x \cup k \cup \zeta_y \cup k'$ bounds a disc. Since k and k' are geodesic segments, the function $d(k(t), k'(\frac{l(k')}{l(k)} t))$ is convex. Therefore we have $d(k(t), k'(\frac{l(k')}{l(k)} t)) \leq 5\epsilon$ for any t since $d(x, h_n(x)) \leq 5\epsilon$ and $d(y, h_n(y)) \leq 5\epsilon$.

We define $g_n : [x, y] \to [h_n(x), h_n(y)]$ by the following formula $g_n([x, y](\frac{l([x,y])}{l(k)} t)) = [h_n(x), h_n(y)](\frac{l([h_n(x), h_n(y)])}{l(k)} t)$. Setting $z = [x, y](\frac{l([x,y])}{l(k)} t)$, the distance $d(z, g_n(z))$ is less than the following quantity

$$d([x, y](\frac{l([x, y])}{l(k)} t), k(t)) + d(k(t), k'(\frac{l(k')}{l(k)} t))$$

$$+d(k'(\frac{l(k')}{l(k)}t,[h_n(x),h_n(y)](\frac{l([h_n(x),h_n(y)])}{l(k)}t)).$$

Then we get $d(z,g_n(z))\leq 5\epsilon+4\epsilon'_n$. Now we conclude by taking N such that $4\epsilon'_n\leq\epsilon$ for $n\geq N$ and by changing A_n by a homotopy so that the geodesic segment $\zeta_n(z)$ connecting z to $g_n(z)$ lies in A_n for any z in $\bar{s}$. $\square$

Now we return to the proof of Lemma 5.1. We remove from $\bar{s}$ its two extremal edges so that g_n is defined on the entire $\bar{s}$ (and we say that $\bar{s}$ originally had $j+2$ edges so that it now has j edges). By the equation (1), there is a constant R' such that for any branch $\hat{b}$ of $\hat{\tau}^1$, we have $l(\tilde{f}_n(\hat{b}))\leq R'\epsilon_n^{-1}$. By Lemma 5.5 and since $\delta_n\longrightarrow 0$ as $n\to\infty$, we have $l_{\rho_n}(g_n(\bar{s}))\leq l_{\rho_n}(\bar{s})+13\epsilon\leq jR'\epsilon_n^{-1}+13\epsilon$. Therefore if we let j'_n be the number of edges contained in $g_n(\bar{s})$, we have

$$j'_n\leq j(\frac{R'}{R})+1 \tag{2}$$

for large n, where R is the constant in (b) of Lemma 4.1.

Since τ^1 has only finitely many branches, once j is fixed, there are only finitely many possibilities for the oriented train routes of $\bar{s}$ and of $g_n(\bar{s})$. Thus we can find $\Theta(p_n)$-many disjoint segments $\{\bar{s}_{1,n},...\}$ with the properties (i), (ii) and (iii), having the same oriented train route, such that the segments $g_n(\bar{s})$ also have the same oriented train route for all $\bar{s}\in\{\bar{s}_{1,n},\dots\}$.

Now we fix j and extract a subsequence so that the oriented train routes of $\bar{s}_{i,n}$ and $g_n(\bar{s}_{i,n})$ do not depend on n.

By the same arguments, we can construct segments containing $j+1$ edges, $j+2$ edges and so on. To avoid difficulties in the next part of the proof, we want to ensure that the segment containing j edges which we have chosen is contained in the segment with $j+1$ edges as its first j segments. For that, to each $\bar{s}_{i,n}$, we add the edge of $\bar{c}_n$ which is adjacent to the last (with respect to the orientation of $\bar{s}_{i,n}$) edge of $\bar{s}_{i,n}$. In this family of segments, take a maximal family of disjoint segments lying in $\bar{c}^1_n$, which we denote by $\{\bar{s}^+_{1,n},\bar{s}^+_{2,n},...\}$. By Claim 5.4 and the argument after that, there are only $o(p_n)$ segments among $\{\bar{s}_{1,n},\dots\}$ for which the added edge lies outside of $\bar{c}^1_n$. Therefore the number of segments in $\{\bar{s}^+_{1,n},\dots\}$ is $\Theta(p_n)$. It follows from the arguments we used for the segments $\bar{s}_{i,n}$ that among the segments $\bar{s}^+_{i,n}$ there are $\Theta(p_n)$ segments which have the properties (i), (ii) and (iii). The proof of Lemma 5.5 applies to these $\Theta(p_n)$ segments, yielding an homeomorphism g_n which can be chosen to coincide with the one defined on the segments $\bar{s}_{i,n}$ when restricted to them.

From this last family, we take $\Theta(p_n)$ segments $\bar{s}^+_{i,n}$ such that the oriented train routes of $\bar{s}^+_{i,n}$ and $g_n(\bar{s}^+_{i,n})$ do not depend on i. Then we extract a subsequence (with respect to n) such that for sufficiently large n, the oriented train routes of $\bar{s}^+_{i,n}$ and of $g_n(\bar{s}^+_{i,n})$ do not depend on n.

By doing this argument recursively, increasing j one by one, we complete the proof of Lemma 5.1. □

6. Commuting elements

In this section, we shall use the results of the preceding sections to construct a sequence of discs which cross λ less and less. This way, we shall obtain a homoclinic simple geodesic which does not cross λ.

Using Lemma 5.1, it is easy to observe that for j (the number of edges that s_n contains) large enough, s_n and s'_n come back simultaneously to a branch of τ_1. Still, the fact that the edges of $\bar{c}^1_n$ are very long makes it difficult to turn this observation into an actual construction. Instead we shall use Lemma 5.1 to construct two sequences a_n, a'_n of elements of $\pi_1(\partial M)$ which have nearly the same actions on some rather large part of $\tilde{f}_n(\hat{\tau}_1)$. By results of Kapovich ([Ka]) this implies that the images of a_n and a'_n in $\pi_1(M)$ commute and therefore correspond to an annulus. Then we shall construct discs as we want by adding some waves.

Recall that given an oriented train route $(bo(i))_{i\in\mathbb{N}}$, we denote by $(b(i))_{i\in\mathbb{N}}$ the corresponding non-oriented train route. Since, by Lemma 5.1, for any t, the simple closed curve c_n goes through the oriented train routes $(bo(i))_{i\leq t}$ and $(bo'(i))_{i\leq t}$ in τ^1 for n large enough, there are two half-leaves l_+ and l'_+ of the realised part of L_{rec} whose oriented train routes are $(bo(i))_{i\in\mathbb{N}}$ and $(bo'(i))_{i\in\mathbb{N}}$ respectively.

Let $e \in \mathbb{N}$ be a natural number which we shall specify later, and let $\varphi : \mathbb{N} \to \mathbb{N}$ be an increasing function such that:

- ($\phi 1$) $bo(\varphi(i) + j) = bo(\varphi(0) + j)$ for any $i \in \mathbb{N}$ and any $0 \leq j \leq e$, i.e. the same train route $(bo(\varphi(0) + j))_{0\leq j\leq e}$ is repeated starting from each $\varphi(i)$.
- ($\phi 2$) $bo'(V(\varphi(i)) + j) = bo'(V(\varphi(0)) + j)$ for any $i \in \mathbb{N}$ and any $0 \leq j \leq V(e)$, where V is the function which appeared in Lemma 5.1.
- ($\phi 3$) Suppose that l_+ (resp. l'_+) is not a closed curve, and let $k_{0,i}$ be a sub-arc of l_+ with train route $(b(j))_{\varphi(0)\leq j<\varphi(i)}$ (resp. $k'_{0,i}$ the arc of l'_+ with train route $(b'(i))_{V(\varphi(0))\leq i<V(\varphi(i))}$). The two endpoints of $k_{0,i}$ (resp $k'_{0,i}$) lie in the same switch of τ^1 and the sequences $(\partial k_{0,i})$ (resp. $\partial k'_{0,i}$) converges to a single point with respect to the Hausdorff topology as $i \longrightarrow \infty$.

The existence of such a function φ follows from the fact that the number of branches and switches of τ^1 is finite. See Figure 5.

We define $\psi : \mathbb{N} \to \mathbb{N}$ as follows: $\psi(i)$ is the largest integer such that $\varphi(\psi(i)) \leq i$. Since φ is not surjective, it does not have an inverse function, and we use ψ instead. Set $\psi_n = \psi(T(n) - e)$ for the function T which appeared in Lemma 5.1. Note that if we forget the condition ($\phi 3$), then $\psi_n + 1$ is the number of times that s_n comes back to the oriented train route $(bo(k))_{\varphi(0)\leq k\leq e+\varphi(0)}$ and s'_n comes back to the oriented train route

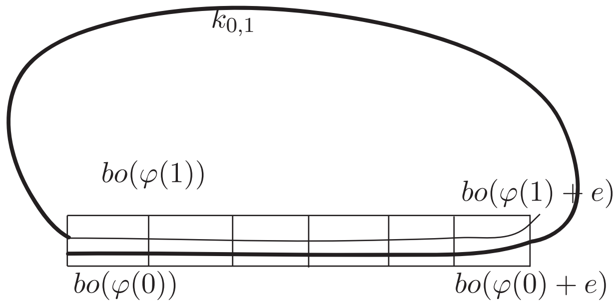

FIGURE 5. Repeating the train route $(bo(\varphi(0)+j))_{0\le j\le e}$.

$(bo'(k))_{V(\varphi(0))\le k\le V(\varphi(0))+V(e)}$ at the same time. Since φ is increasing and T is unbounded, (ψ_n) is unbounded.

Let us denote by b' the branch $bo'(V(\varphi(0)))$. By Lemma 5.1, for any $i\le \psi_n$, we have $g_n\circ f_n\circ v(\varphi(i))\in f_n(b'(V(\varphi(i)))=f_n(b')$. It follows that, for any n large, there are $i_n, j_n\le\psi_n$ such that the distance between $g_n\circ f_n\circ v(\varphi(i_n))$ and $g_n\circ f_n\circ v(\varphi(j_n))$ measured on $f_n(b')$ is at most $\frac{1}{\psi_n}l_{\rho_n}(f_n)(b')=o(\epsilon_n^{-1})$. Allowing $g_n\circ f_n\circ v(\varphi(i_n))$ and $g_n\circ f_n\circ v(\varphi(j_n))$ to be a bit further from each other, we can still keep their distance to be $o(\epsilon_n^{-1})$ while assuming that $j_n-i_n\longrightarrow\infty$.

We pick such sequences (i_n) and (j_n), i.e. for each n, we take two indices $i_n<j_n\le\psi_n$ such that $j_n-i_n\longrightarrow\infty$ and the distance between $g_n\circ f_n\circ v(\varphi(i_n))$ and $g_n\circ f_n\circ v(\varphi(j_n))$ measured on $f_n(b')$ is $o(\epsilon_n^{-1})$. We denote by $I_n\subset s_n$ the segment between $v(\varphi(i_n))$ and $v(\varphi(j_n))$ and by J_n be the sub-segment of s_n consisting of the e vertices following $v(\varphi(j_n))$.

Let $\tilde s_n\subset\mathbb{H}^3$ be a lift of $f_n(s_n)$, and let $\tilde v(\varphi(i_n))$, $\tilde v(\varphi(j_n))$, $\tilde I_n$ and $\tilde J_n$ be lifts of $f_n\circ v(\varphi(i_n))$, $f_n\circ v(\varphi(j_n))$, $f_n(I_n)$ and $f_n(J_n)$ respectively, lying on $\tilde s_n$. We lift the map g_n to a map $\tilde g_n$ from $\tilde s_n$ to a lift $\tilde s'_n$ of $f_n(s'_n)$. Let $\rho_n(a_n)\in\rho_n(\pi_1(M))$ be the covering translation which takes $\tilde f_n(\tilde b(\varphi(i_n))$ to $\tilde f_n(\tilde b(\varphi(j_n))$. Since $bo(\varphi(i_n)+j)=bo(\varphi(j_n)+j)$ for all $j\le e$, the isometry $\rho_n(a_n)$ acts as a translation on $\tilde I_n\cup\tilde J_n$.

Let $\tilde I'_n\subset\tilde s'_n$ be the piecewise geodesic segment between $\tilde g_n\circ\tilde v(\varphi(i_n))$ and $\tilde g_n\circ\tilde v(\varphi(j_n))$, and let $\tilde J'_n\subset\tilde s'_n$ be the segment between $\tilde g_n\circ\tilde v(\varphi(i_n))$ and $\tilde g_n\circ\tilde v(\varphi(j_n)+e)$. Let $\rho_n(a'_n)\in\rho_n(\pi_1(M))$ be the covering translation which takes $\tilde f_n(\tilde b'(V(\varphi(i_n)))$ to $\tilde f_n(\tilde b'(V(\varphi(j_n)))$. From the assumption that $bo'(V(\varphi(i_n))+j)=bo'(V(\varphi(j_n))+j)$ for any $0\le j\le V(e)$, it follows that $\rho_n(a'_n)$ acts as a translation on $\tilde I'_n\cup\tilde J'_n$. See Figure 6.

By our choice of i_n, j_n, we have

$$d(\tilde g_n\circ\tilde v(\varphi(j_n)),\rho_n(a'_n)\circ\tilde g_n\circ\tilde v(\varphi(i_n))=o(\epsilon_n^{-1}).$$

From this together with the facts that $\rho_n(a_n)$ acts as a translation on $\tilde I_n\cup\tilde J_n$ and that $\rho_n(a'_n)$ acts as a translation on $\tilde I'_n\cup\tilde J'_n$, we shall deduce the following claim:

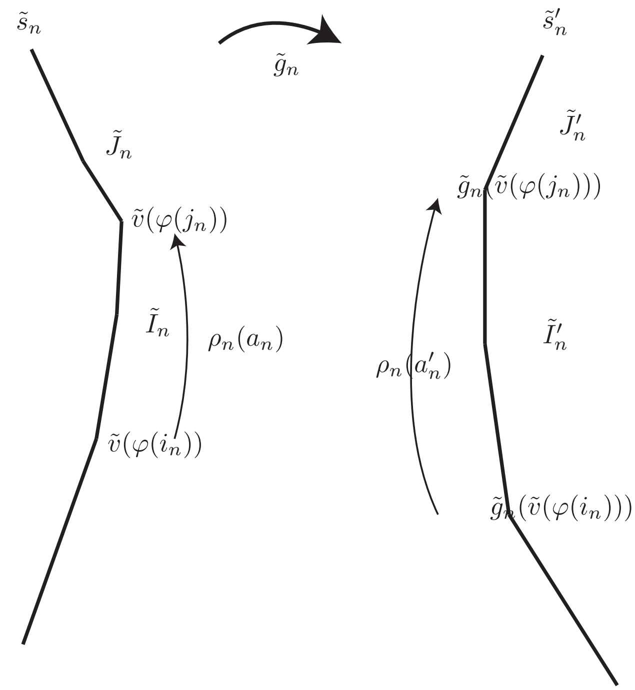

FIGURE 6. Commuting isometries

Claim 6.1. *For $R > 0$, let $\mathcal{V}_R(\tilde{J}_n)$ be the R-neighbourhood of $\tilde{J}_n$, then for any sequence of points $\tilde{z}_n \in \mathcal{V}_R(\tilde{J}_n)$, we have that $d(\tilde{z}_n, \rho_n(a_n^{-1}a'_n)(\tilde{z}_n))$ is $o(\epsilon_n^{-1})$.*

Proof. It is sufficient to prove this claim for any sequence $(\tilde{z}_n)$ lying on $\tilde{J}_n$. Since $\rho_n(a_n^{-1})$ acts as a translation on $\tilde{I}_n \cup \tilde{J}_n$, the point $\rho_n(a_n^{-1})(\tilde{z}_n)$ is the point $\tilde{z}'_n \in \tilde{I}_n$ with

$$d(\tilde{z}_n, \tilde{v}(\varphi(j_n))) = d(\tilde{z}'_n, \tilde{v}(\varphi(i_n))), \tag{3}$$

where d denotes the distance measured on $\tilde{I}_n \cup \tilde{J}_n$. The point $\tilde{z}''_n = \rho_n(a'_n) \circ \tilde{g}_n(\tilde{z}'_n) \in \tilde{J}'_n$ is at the distance $d(\tilde{g}_n(\tilde{z}'_n), \tilde{g}_n \circ \tilde{v}(\varphi(i_n))$ from $\rho_n(a'_n) \circ \tilde{g}_n \circ \tilde{v}(\varphi(i_n))$ (measured on $\tilde{I}'_n \cup \tilde{J}'_n$). As we saw above, $d(\tilde{g}_n \circ \tilde{v}(\varphi(j_n)), \rho_n(a'_n) \circ \tilde{g}_n \circ \tilde{v}(\varphi(i_n))) = o(\epsilon_n^{-1})$. Therefore we have $d(\tilde{z}''_n, \tilde{g}_n \circ \tilde{v}(\varphi(j_n))) = d(\tilde{g}_n(\tilde{z}'_n), \tilde{g}_n \circ \tilde{v}(\varphi(i_n))) + o(\epsilon_n^{-1})$.

By Lemma 5.1, $\tilde{g}_n$ moves each point within distance 6ϵ, and hence we get the equality

$$d(\tilde{z}''_n, \tilde{g}_n \circ \tilde{v}(\varphi(j_n))) = d(\tilde{z}'_n, \tilde{v}(\varphi(i_n))) + o(\epsilon_n^{-1}).$$

Using equation 3, we get

$$d(\tilde{z}''_n, \tilde{g}_n \circ \tilde{v}(\varphi(j_n))) = d(\tilde{z}_n, \tilde{v}(\varphi(j_n)) + o(\epsilon_n^{-1}).$$

Since $\tilde{J}_n$ and $\tilde{J}'_n$ are within the Hausdorff distance 6ϵ from each other, we have

$$d(\tilde{z}''_n, \tilde{z}_n) = o(\epsilon_n^{-1}).$$

By the triangle inequality

$$d(\rho_n(a'_n a_n^{-1})(\tilde{z}_n), \tilde{z}_n) \leq d(\tilde{z}_n, \tilde{z}''_n) + d(\tilde{z}''_n, \rho_n(a'_n a_n^{-1})(\tilde{z}_n)).$$

and by $\tilde{z}''_n = \rho_n(a'_n) \circ \tilde{g}_n \circ \rho_n(a_n^{-1})(\tilde{z}_n)$ we get

$$\begin{aligned} d(\tilde{z}''_n, \rho_n(a'_n a_n^{-1})\tilde{z}_n) &= d(\rho_n((a'_n)^{-1})\tilde{z}''_n, \rho_n(a_n^{-1})\tilde{z}_n) \\ &= d(\tilde{g}_n \circ \rho_n(a_n^{-1})(\tilde{z}_n), \rho_n(a_n^{-1})(\tilde{z}_n)) = o(\epsilon_n^{-1}). \end{aligned}$$

Thus we finally get $d(\rho_n(a'_n a_n^{-1})(\tilde{z}_n), \tilde{z}_n) = o(\epsilon_n^{-1})$. □

We note that, as can be seen in the proof, the $o(\epsilon_n^{-1})$ is "uniform", namely there is a sequence $\delta_n \longrightarrow 0$ independent of $(\tilde{z}_n)$ such that $d(\rho_n(a'_n a_n^{-1})(\tilde{z}_n), \tilde{z}_n) \leq \delta_n \epsilon_n^{-1}$.

We shall use this claim to prove the following lemma.

Lemma 6.2. *There is $N \in \mathbb{N}$ such that for $n \geq N$, $\rho_n(a_n^{-1}a'_n) = id$.*

The main argument in the proof is the following.

Lemma 6.3. *Let $[P_n, Q_n] \subset \mathbb{H}^3$ be a sequence of geodesic segments between P_n and Q_n such that $l([P_n, Q_n])$ is $\Theta(\epsilon_n^{-1})$ and let $(\delta_n), (\delta'_n) \subset \pi_1(M)$ be two sequences such that the distances $d(P_n, \rho_n(\delta_n)(P_n))$, $d(P_n, \rho_n(\delta'_n)(P_n))$, $d(Q_n, \rho_n(\delta_n)(Q_n))$ and $d(Q_n, \rho_n(\delta'_n)(Q_n))$ are all $o(\epsilon_n^{-1})$. Then there is N such that for $n \geq N$, $[\rho_n(\delta_n), \rho_n(\delta'_n)] = id$.*

Proof. This comes directly from the arguments which Kapovich used in [Ka, Theorem 10.24] to prove that the minimal action of $\pi_1(M)$ on $\mathbb{R}$-tree is small (see [Ka, p.239]). □

Proof of Lemma 6.2. Set $e = 2p + 1$ where p is the number of the branches of τ^1. If we fix some $n \in \mathbb{N}$, by our choice of e, we can find two different integers i_1 and i_2 between $\varphi(j_n)$ and $\varphi(j_n) + e$ such that $bo(i_1) = bo(i_2)$. Let $K_n \subset J_n$ be the segment of s_n with train route $(b(k))_{i_1 \leq k \leq i_2}$, and let $\delta_n \in \pi_1(M)$ be an element which takes $\tilde{v}(i_1)$ to $\tilde{v}(i_2)$. Since i_1, i_2 do not depend on n, the element δ_n does not depend on n either. Let us denote it by g. The isometry $\rho_n(g)$ acts as a translation on the lift $\tilde{K}_n$ of $f_n(K_n)$ which lies in $\tilde{s}_n$.

Let $\tilde{K}_{n-}$ and $\tilde{K}_{n+}$ be the two extremal edges of $\tilde{K}_n$, such that $\rho_n(g)$ takes $\tilde{K}_{n-}$ to $\tilde{K}_{n+}$. By Claim 6.1, the translation length of $\rho_n(a_n^{-1}a'_n)$ is $o(\epsilon_n^{-1})$ on $\tilde{I}_n \cup \tilde{J}_n$. By the same arguments, the translation length of $\rho_n(a'^{-1}_n a_n)$ is also $o(\epsilon_n^{-1})$ on $\tilde{I}_n \cup \tilde{J}_n$. It follows that:

- for any sequence $\tilde{z}_n \in \mathcal{V}_R(\tilde{K}_{n+})$, $d(\tilde{z}_n, [\rho_n(a_n^{-1}a'_n), \rho_n(g)](\tilde{z}_n)) = o(\epsilon_n^{-1})$;

- for any sequence $\tilde{z}_n \in \mathcal{V}_R(\tilde{K}_{n-})$, $d(\tilde{z}_n, [\rho_n(g^{-1}), \rho_n(a_n^{-1}a'_n)](\tilde{z}_n)) = o(\epsilon_n^{-1})$;

- for any sequence $\tilde{z}_n \in \mathcal{V}_R(\tilde{K}_{n+})$, $d(\tilde{z}_n, [\rho_n(g), \rho_n(a_n'^{-1}a_n)](\tilde{z}_n)) = o(\epsilon_n^{-1})$.

Since $\tilde{K}_{n+}$ and $\tilde{K}_{n-}$ are edges of $\bar{c}_n^1$, their lengths are $\Theta(\epsilon_n^{-1})$. Applying Lemma 6.3 to the segments $\tilde{K}_{n-}$ and $\tilde{K}_{n+}$, we see that for sufficiently large n, $[\rho_n(a_n^{-1}a'_n), \rho_n(g)]$, $[\rho_n(g^{-1}), \rho_n(a_n^{-1}a'_n)]$ and $[\rho_n(g), \rho_n(a_n'^{-1}a_n)]$ commute with $\rho_n(a_n^{-1}a'_n)$. Therefore they belong to an elementary subgroup of $\rho_n(\pi_1(M))$. By [Ka, p.239], it follows that the group generated by $\rho_n(a_n^{-1}a'_n)$ and $\rho_n(g)$ is elementary.

Since $i_1 \neq i_2$, the distance of translation of $\rho_n(g)$ is $\Theta(\epsilon_n^{-1})$. Since the group generated by $\rho_n(a_n^{-1}a'_n)$ and $\rho_n(g)$ is elementary, there are $d \in \pi_1(M)$, $t, t_n \in \mathbb{N}$ such that $g = d^t$ and $a_n^{-1}a'_n = d^{t_n}$. Since the translation distance of $\rho_n(g)$ is $\Theta(\epsilon_n^{-1})$, so is the translation distance of $\rho_n(d)$. By Lemma 6.1 however, the translation distance of $\rho_n(a_n^{-1}a'_n)$ is $o(\epsilon_n^{-1})$. Therefore we have $\rho_n(a_n^{-1}a'_n) = id$ for sufficiently large n. □

Now we are ready to construct the homoclinic geodesic which was announced at the beginning of the section.

Lemma 6.4. *Under the light assumptions, assume that $l_{\rho_n}(\lambda_n^*)$ is bounded, then there is a homoclinic simple geodesic which does not cross $|\lambda|$.*

Proof. Recall that, earlier in this section, we have picked a lift $\tilde{s}_n \subset \mathbb{H}^3$ of $f_n(s_n)$, and lifts $\tilde{v}(\varphi(i_n))$, $\tilde{v}(\varphi(j_n))$ of $f_n \circ v(\varphi(i_n))$ and $f_n \circ v(\varphi(j_n))$ respectively, lying on $\tilde{s}_n$. We have also lifted the map g_n to a map $\tilde{g}_n$ from $\tilde{s}_n$ to a lift $\tilde{s}'_n$ of $f_n(s'_n)$ and we have defined $a_n, a'_n \in \pi_1(M)$.

To simplify the notations, set $\tilde{x}_n = \tilde{v}(\varphi(i_n)$ and $\tilde{y}_n = \tilde{v}(\varphi(j_n)$. Recall that $\tilde{y}_n = \rho_n(a_n)(\tilde{x}_n)$. By Lemma 6.2, $\rho_n(a_n)(\tilde{g}_n(\tilde{x}_n)) = \rho_n(a'_n)(\tilde{g}_n(\tilde{x}_n))$, which implies that $\rho_n(a_n)(\tilde{g}_n(\tilde{x}_n))$ lies on the same component of $\bar{c}_n^1$ as $\tilde{g}_n(\tilde{y}_n)$. It follows that we can change A_n so that $\rho_n(a_n)(\tilde{\zeta}_n(\tilde{x}_n))$ (recall that $\tilde{\zeta}_n(x_n)$ is a short segment in $\tilde{A}_n$ which joins $\tilde{x}_n$ to $\tilde{g}_n(\tilde{x}_n)$) lies in $\tilde{A}_n$. For the sake of simplicity, we set $\tilde{\zeta}_n(\tilde{y}_n) = \rho_n(a_n)(\tilde{\zeta}_n(\tilde{x}_n))$.

Denote by $\tilde{k}_n$ the segment of $\tilde{c}_n$ which is mapped by $\tilde{f}_n$ to an arc joining $\tilde{x}_n$ to $\tilde{y}_n$, namely $\partial \tilde{f}_n(\tilde{k}_n) = \{\tilde{x}_n, \tilde{y}_n\}$. Similarly denote by $\tilde{k}'_n$ the segment of $\tilde{c}_n$ which is mapped by $\tilde{f}_n$ to an arc joining $\tilde{\zeta}_n(\tilde{x}_n)$ to $\tilde{\zeta}_n(\tilde{y}_n)$. By construction, $\tilde{k}_n$ and $\tilde{k}'_n$ lie in lifts of s_n and s'_n respectively and goe through the train routes $(bo(i))_{\varphi(i_n) \leq i < \varphi(j_n)}$ and $(bo'(i))_{V(\varphi(i_n)) \leq i < V(\varphi(j_n))}$ respectively.

The arcs $\tilde{f}_n(\tilde{k}_n)$, $\tilde{f}_n(\tilde{k}'_n)$, $\tilde{\zeta}_n(\tilde{x}_n)$ and $\tilde{\zeta}_n(\tilde{y}_n)$ bound a disc $\tilde{R}_n$ in $\tilde{A}_n$ (by Lemma 5.1) and all of their endpoints lie on the same component $\tilde{f}_n(\tilde{c}_n)$ of $\partial \tilde{A}_n$. It follows that one component of $\tilde{A}_n - \tilde{R}_n$ is a disc $\tilde{D}_n$. The boundary of this disc is the union of an arc $\tilde{d}_n \subset \tilde{f}_n(\tilde{c}_n)$ and $\tilde{\zeta}_n(\tilde{x}_n)$ or $\tilde{\zeta}_n(\tilde{y}_n)$. See Figure 7.

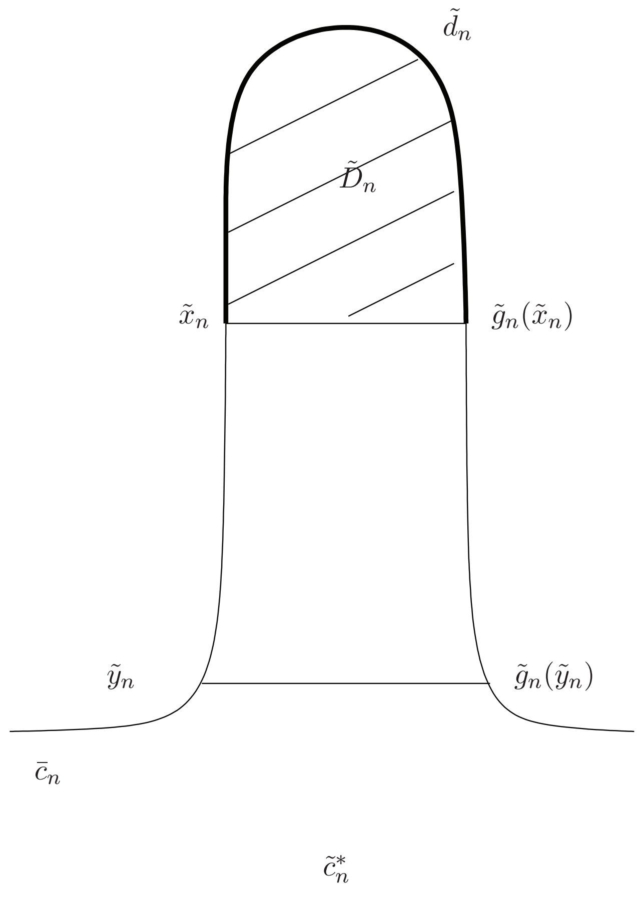

FIGURE 7. The innermost disc $\tilde{D}_n$

Let us fix a reference hyperbolic metric on ∂M. The endpoints of k_n (resp. k'_n) are connected by an arc κ_n (resp. κ'_n) which lies on a switch of τ^1. By Lemma 6.2, the closed curves $k_n \cup \kappa_n$ and $k'_n \cup \kappa'_n$ are homotopic in M. In particular they bound an annulus E_n in M.

Consider the projection $d_n \subset c_n$ of the arc which is mapped to $\tilde{d}_n$ by $\tilde{f}_n$. The closed curve $m_n = \kappa_n k_n d_n k_n'^{-1} \kappa_n'^{-1} d_n^{-1}$ (with appropriate choices of orientation) bounds a (possibly singular) disc which is the union of E_n and two copies of the projection D_n of $\tilde{D}_n$ (see Figure 8).

Since both $j_n - i_n$ and i_n can be assumed to go to ∞, it follows from the property ($\phi 3$) that the length of κ_n tends to 0 (with respect to our reference metric on ∂M). It follows that $\int_{m_n} d\lambda_n \longrightarrow 0$. By the proof of the Loop Theorem (see [He] for example), there is a meridian m'_n such that $i(m'_n, \lambda_n) \longrightarrow 0$. By Casson's criterion (Theorem 2.4) this concludes the proof of Lemma 6.4. $\square$

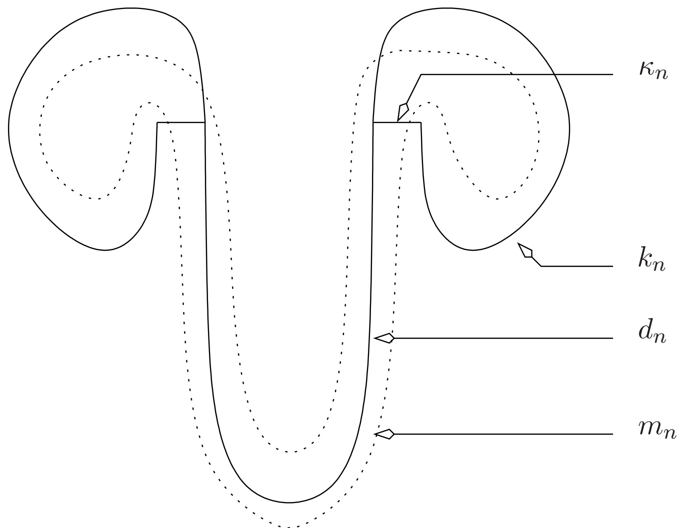

FIGURE 8. The curve m_n

7. CONCLUSION

In this section, we shall complete the proofs of Theorem 2 and 1.

Proof of Theorem 2. Now we complete the proof of Theorem 2. Let ρ_n, λ_n and λ be as in Theorem 2. If no subsequence of (ρ_n) converges algebraically, it follows from Lemma 4.1 that ρ_n and λ_n satisfy the light assumptions. Moreover $l_{\rho_n}(\lambda_n)$ is bounded by assumption. By Lemma 6.4, there is a homoclinic geodesic which does not intersect $|\lambda|$ transversely. By Lemma 2.6, this contradicts the assumption that $\lambda \in \mathcal{D}(M)$. □

We shall now deduce Theorem 1 from Theorem 2.

Theorem 1. *Let M be a compact irreducible atoroidal 3-manifold with boundary. Let (m_n) be a sequence in the Teichmüller space $\mathcal{T}(\partial M)$ which converges in the Thurston compactification to a projective lamination $[\lambda]$ contained in $\mathcal{PD}(M)$. Let $q : \mathcal{T}(\partial M) \to GF_0(M,P)$ be the Ahlfors-Bers map, and suppose that $(\rho_n : \pi_1(M) \to PSL(2,\mathbb{C}))$ is a sequence of discrete faithful representations corresponding to $(q(m_n))$. Then passing to a subsequence, (ρ_n) converges in $AH(N)$.*

Proof. For a simple closed curve $c \subset \partial M$, we denote by $l_{m_n}(c)$ the length of c with respect to the metric m_n and by $l_{\rho_n}(c)$ the length of the closed geodesic of $\mathbb{H}^3/\rho_n(\pi_1(M))$ in the free homotopy class of c. By [Th2, Theorem 2.2] (see also [FLP]), there is a sequence of simple closed curves $c_n \subset \partial M$ whose projective classes converge to $[\lambda]$ in $\mathcal{PML}(\partial M)$ such that $\dfrac{l_{m_n}(c_n)}{l_{m_0}(c_n)}$ tends to 0 as $n \to \infty$.

Using the following result of [BrC], we shall get an upper bound for the sequence $(l_{\rho_n}(\lambda_n))$.

Theorem C [Bridgeman-Canary] *For any $Q > 0$, there is a constant $K > 0$ depending only on Q with the following properties. Let Γ be a finitely generated Kleinian group without torsion such that the shortest length of the meridians with respect to the compatible hyperbolic structure on Ω_Γ/Γ is greater than Q. Let $C(\Gamma)$ be the convex core of $\mathbb{H}^3/\Gamma$, and consider the nearest point retraction $r : \Omega_\Gamma/\Gamma \to \partial C(\Gamma)$. Then r is K-Lipschitz and has a homotopically inverse K-Lipschitz map.*

Let us first verify that we are considering a situation where the hypothesis of this theorem is fulfilled.

Lemma 7.1. *There is a positive number Q such that $l_{m_n}(d) \geq Q$ for any meridian d of ∂M.*

Proof. Assuming the contrary, we have a sequence of meridians (d_n) such that $(l_{m_n}(d_n))$ tends to 0. Let us extract a subsequence so that (d_n) converges with respect to the Hausdorff topology to a geodesic lamination $D \subset M$. By Casson's criterion, D contains a homoclinic leaf. Since $[\lambda] \in \mathcal{PD}(M)$, the lamination D intersects the support of $[\lambda]$ transversely (Lemma 2.6). It follows that the sequence $i(\lambda, d_n)$ is bounded away from 0. This implies that the sequence $l_{m_n}(d_n)$ tends to ∞. Thus we get a contradiction. $\square$

It is clear that the length $l_{\rho_n}(c_n^*)$ is less than the length of any curve in $\partial C(\rho_n(\pi_1(M)))$ which is freely homotopic to c_n. Thus, applying Theorem C, we see that $\dfrac{l_{\rho_n}(c_n^*)}{l_{m_0}(c_n)}$ tends to 0.

Let us denote by λ_n the measured geodesic lamination obtained by endowing c_n with a Dirac measure whose weight is equal to $l_{m_0}(c_n)^{-1}$. The sequence λ_n converges in $\mathcal{ML}(\partial M)$ to a measured geodesic lamination λ which lies in the projective class $[\lambda]$. Since $\dfrac{l_{\rho_n}(c_n^*)}{l_{m_0}(c_n)}$ tends to 0, we have $l_{\rho_n}(\lambda_n) \longrightarrow 0$. Since λ lies in the projective class $[\lambda] \in \mathcal{P}D(M)$, the measured geodesic lamination λ lies in $\mathcal{D}(M)$. Applying Theorem 2, we see that a subsequence of (ρ_n) converges algebraically. $\square$

8. Appendix

For the sake of completeness, we shall give brief proofs of some propositions which we cited in previous sections.

Proposition 2.2 [[Le1], §2 paragraphs after Lemme 2.7] *Let Σ and $\Sigma' \subset \partial_{\chi<0}M$ be two compact, connected, incompressible surfaces which are disjoint or equal and do not contain any essential closed curve which can be homotoped into $\partial_{\chi=0}M$. Let $\tilde\Sigma \subset \partial\tilde M$ (resp. $\tilde\Sigma'$) be a connected component* *of the preimage of* Σ *(resp.* Σ'*) and let* $\Gamma \subset \rho(\pi_1(M))$ *(resp.* Γ'*) be the stabiliser of* $\tilde{\Sigma}$ *(resp.* Γ'*).*

Then $\bar{\tilde{\Sigma}} \cap \bar{\tilde{\Sigma}}'$ *is either empty or equal to the limit set of* $\Gamma \cap \Gamma'$*.*

In the latter case, if $\Gamma \cap \Gamma'$ *is not cyclic, then it is the fundamental group of a (possibly twisted)* I*-bundle which is a connected component of a characteristic submanifold of* $(M, \Sigma \cup \Sigma')$*. If* $\Gamma \cap \Gamma'$ *is cyclic, then it is a finite index subgroup of a solid torus which is a connected component of a characteristic submanifold of* $(M, \Sigma \cup \Sigma')$*.*

Proof. Suppose that ξ belongs to the limit set of $\Gamma \cap \Gamma'$. Since we are considering only geometrically finite groups, both Γ and Γ' are convex cocompact. Let l be a geodesic ray from the origin $O \in \mathbb{H}^3$ to ξ. Let F be a fundamental domain of the convex core of Γ containing O, and F' a fundamental domain of the convex core of Γ' containing O. Since Γ and Γ' are convex cocompact, the diameters of F and F' are bounded. Choose $g_n \in \Gamma, g'_n \in \Gamma'$ such that $g_n F \cap l \neq \emptyset, g'_n F' \cap l \neq \emptyset$ and $g_n F \cap g'_n F' (\neq \emptyset) \longrightarrow \xi$. Since the diameters of F and F' are bounded, we have

$$d(g_n O, g'_n O) = d(O, g_n^{-1} g'_n O) \leq K$$

for all n and a fixed K. By discreteness of $\rho(\pi_1(M))$, after passing to a subsequence, $g_n^{-1} g'_n = g$ for all n. Then $g_n^{-1} g'_n = g_m^{-1} g'_m$, i.e., $g_m g_n^{-1} = g'_m g'^{-1}_n \in \Gamma \cap \Gamma'$. Since Γ and Γ' are convex cocompact, these elements are hyperbolic and the limit set of $\Gamma \cap \Gamma'$ contains at least two points. Let $h \in \Gamma \cap \Gamma'$ be a hyperbolic element. Then the invariant geodesics $\tilde{c}, \tilde{c}'$ of h in $\tilde{\Sigma}, \tilde{\Sigma}'$ descend to closed geodesics c and c' in Σ and Σ'. Hence c and c' bound an annulus A (not necessarily embedded) which is not homotoped into ∂M. Hence A must be contained in some characteristic submanifold of $(M, \Sigma \cup \Sigma')$. If $\Sigma = \Sigma'$, then $c = c'$ and A is a Möbius band.

Let $F = \bar{\tilde{\Sigma}} \cap \bar{\tilde{\Sigma}}'$ and C, C' be the projections of the convex hulls of F in $\tilde{\Sigma}$ and $\tilde{\Sigma}'$. Then $\pi_1(C) = \Gamma \cap \Gamma'$ and $C \cup C'$ is the boundary of an I-bundle $C \times I$ which is essential in $(M, \Sigma \cup \Sigma')$. If $\Sigma = \Sigma'$, then it is a twisted I-bundle with the boundary C. □

Next we include some proofs from [KlS] for the reader's convenience.

Proposition 8.1. *Suppose that the action of* $\pi_1(M)$ *on an* $\mathbb{R}$*-tree* $\mathcal{T}$ *is minimal and small. Let* $S \subset \partial M$ *be a compact compressible surface which has (possibly empty) geodesic boundary with respect to a hyperbolic metric on* ∂M*. Suppose that* $\phi : \mathcal{T}_\mu \to \mathcal{T}$ *is a* $\pi_1(M)$*-equivariant morphism which folds only at complementary regions. If* $\mu \subset S$ *is in tight position with respect to a meridian* $m \subset S$*, then* $|\mu|$ *can be extended to a geodesic lamination which contains a homoclinic leaf* h *in* S*.*

Proof. Fix a hyperbolic metric on ∂M. The universal cover $\tilde{S} \subset \mathbb{H}^2$ is a convex subset of $\mathbb{H}^2$. Let $\bar{m}$ represent an element of $\pi_1(S)$ corresponding to the meridian m, which leaves invariant a lift $\tilde{m} \subset \tilde{S}$. By equivariance, for

$x \in \tilde{m}$,

$$\phi(\pi(\bar{m}x)) = \phi(\pi(x))$$

since $\bar{m}$ is trivial in $\pi_1(M)$, where $\pi : \tilde{S} \to \mathcal{T}_\mu$ is an equivariant map. Since ϕ folds $[\pi(x), \pi(\bar{m}x)]$ only finitely many times, one can find segments $\tilde{I}_1, \tilde{I}_2 \subset \tilde{m}$ such that $\tilde{I}_1 \cap \tilde{I}_2 = y \in \tilde{m}$ and ϕ folds $\pi(\tilde{I}_1)$ and $\pi(\tilde{I}_2)$ along $\pi(y)$. For $x \in m \cap \mu$, let μ_x^+ denote a half leaf of μ starting from x to a chosen positive direction. Then using Skora's idea, Kleineidam and Souto showed [KlS] (Proposition 3) that there are $z_i \in I_i \cap \mu$ such that the lifts of $\mu_{z_1}^+$ and $\mu_{z_2}^+$ to $\partial\tilde{M}$ have the same endpoints. Let C be the complementary region of $\tilde{\mu}$ in $\tilde{S}$, which contains the folding point y. Let $\tilde{\mu}_1, \tilde{\mu}_2$ be boundary leaves of C. Up to reversing the orientation, we can assume that $\tilde{\mu}_1^+, \tilde{\mu}_2^+$ are not asymptotic in $\tilde{S}$. Since the lifts of $\mu_{z_1}^+$ and $\mu_{z_2}^+$ to $\partial\tilde{M}$ have the same end points, by shrinking the intervals, we can see that the projection of $\tilde{\mu}_1^+, \tilde{\mu}_2^+$ to $\partial\tilde{M}$ have the same endpoint. Let l be the geodesic in $\tilde{S}$ joining the end points of $\tilde{\mu}_1^+, \tilde{\mu}_2^+$. The projection of l to S becomes a homoclinic leaf disjoint from μ. □

Proposition 8.2. *Let S be a compressible surface in ∂M, which contains a homoclinic leaf h. Then there is a sequence of meridians whose Hausdorff limit does not cross h.*

Proof. Since a homoclinic leaf cannot be contained in an incompressible surface by Lemma 2.1, $S(\bar{h})$ must contain a meridian m.

If h contains infinitely many homotopy classes of m-waves. Then there are $(x_i), (y_i) \subset \mathbb{R}$ such that $h(x_i), h(y_i) \in m$ and $h[x_i, y_i]$ are non-homotopic m-waves. Since m is compact, after passing to a subsequence, we may assume that $h(x_i)$ and $h(y_i)$ converge. Hence for any $\epsilon > 0$, we can choose i, j such that the lengths of segments $[h(x_i), h(x_j)], [h(y_i), h(y_j)] \subset m$ are less than ϵ. Then $h[x_i, y_i] \cup h[x_j, y_j] \cup [h(x_i), h(x_j)] \cup [h(y_i), h(y_j)]$ is a meridian whose geodesic representative lies nearby the homoclinic leaf.

If h contains only finitely many homotopy classes of m-waves. then there is a meridian m and two disjoint half-leaves h^+ and h^- of h such that h^+ and h^- are in tight position with respect to m. Considering the intersections of m and h^+ and h^- respectively, one obtains a picture similar to Figure 8, namely there is an arc $k \subset h^+$ and an arc $k' \subset h^-$ which nearly bounds an annulus and a wave between k and k'. These arcs can be used to construct a sequence of meridians whose Hausdorff limit does not cross h as explained in the proof of [KlS, Proposition 1]. □

With more work, one could probably prove that $\bar{h}$ is a Hausdorff limit of meridians. On the other hand, in all the situations we have used Proposition 8.2, with only little changes, we could have replaced it with the following weaker result whose proof is easier.

Lemma 8.3. *Let S be a compressible surface which contains a homoclinic leaf h, and let β be a measured lamination which does not cross h. Then there is a sequence of meridians whose Hausdorff limit does not cross β.*

Proof. Using cut-and-paste operation as in Claim 2.3, we construct a sequence of meridians m_i such that $i(m_i, \beta) \longrightarrow 0$. Start with a meridian $m \subset S(\bar{h})$. Since h is homoclinic, it contains an m-wave. Using this wave as in the proof of Claim 2.3, we get a meridian m_1 such that $i(m_1, \beta) \leq \frac{1}{2} i(m, \beta)$. Then we do the same again on m_1. Repeating this, we get a sequence of meridians m_i such that $i(m_i, \beta) \leq \frac{1}{2^i} i(m, \beta)$. □

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