An extension of the Masur domain

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Abstract

The Masur domain is a subset of the space of projective measured geodesic laminations on the boundary of a 3-manifold M. This domain plays an important role in the study of the hyperbolic structures on the interior of M. In this paper, we define an extension of the Masur domain and explain that it shares a lot of properties with the Masur domain.

1 Introduction

A compression body is the connected sum along the boundary of a ball of I-bundles over closed surfaces and solid tori. Among the compression bodies are the handlebodies which are the connected sums along the boundary of solid tori $D^2 \times S^1$. If M is a compression body and if ∂M has negative Euler characteristic then, by Thurston hyperbolization theorem, its interior admits a hyperbolic structure. Namely there are discrete faithful representations $\rho : \pi_1(M) \to Isom(\mathbb{H}^3)$ such that $\mathbb{H}^3/\rho(\pi_1(M))$ is homeomorphic to the interior of M. If such a representation ρ is geometrically finite, it is said to uniformize M.

In [Ma], H. Masur studied the space of projective measured foliations on the boundary of a handlebody. He described the limit set of the action of the modular group on this space and defined a subset of the space of projective measured foliations on which this action is properly discontinuous. In [Ot1], J.-P. Otal defined a similar subset $\mathcal{O}$ of the space of projective measured geodesic laminations on the boundaries of compression bodies. This set $\mathcal{O} \subset \mathcal{PML}(\partial M)$ is called the Masur domain and J.-P. Otal showed that the action of the modular group on $\mathcal{O}$ is properly discontinuous. He also proved the following : if $int(M)$ is endowed with a convex cocompact hyperbolic metric, then any projective class of measured geodesic laminations lying in $\mathcal{O}$ is realized by a pleated surface. He also showed that the injectivity theorem of [Th2] applies for such pleated surfaces.

Later it was shown that the projective classes of measured laminations in $\mathcal{O}$ are an analogous of what Thurston called binding laminations on I-bundles over closed surfaces. Namely if we have a sequence of geometrically finite representations $\rho_n : \pi_1(M) \to Isom(\mathbb{H}^3)$ uniformizing a compression body and a measured geodesic lamination $\lambda \in \mathcal{O}$ such that $l_{\rho_n}(\lambda)$ is bounded, then the sequence (ρ_n) contains an algebraically converging subsequence. This property has been obtained for various cases in [Th3], [Ot2], [Ca], [Oh2] and the general statement comes from [KlS1] and [KlS2].

In this paper, we allow M to be any orientable 3-manifold with boundary satisfying the following : the Euler characteristic of ∂M is negative and the interior of M admits a complete hyperbolic metric. We will consider the following set :

$\mathcal{D}(M) = \{\lambda \in \mathcal{ML}(\partial M) | \, \exists \eta > 0$ such that $i(\lambda, \partial E) > \eta$ for any essential annulus or disc $E \subset M\}$.

First we will link this set $\mathcal{D}(M)$ with the result of [Le1] and deduce from this that the support of a geodesic measured lamination lying in $\mathcal{D}(M)$ is also the support of a (in fact many) bending measured geodesic lamination of a representation uniformizing M. Using the continuity of the bending measure proved in [KeS] and [Bo2], we will show that $\mathcal{D}(M)$ is connected. It follows from the ideas of [Ot1] that the projection of $\mathcal{D}(M)$ on $\mathcal{PML}(\partial M)$ contains $\mathcal{O}$ and we will use this to show that the Masur domain is connected.

After that, we will prove that the set $\mathcal{D}(M)$ has the following properties :

If $int(M)$ is endowed with a convex cocompact hyperbolic metric, any measured geodesic lamination lying in $\mathcal{D}(M)$ is realized by a pleated surface and such a pleated surface satisfies the injectivity theorem of [Th2].

If ρ_n is a sequence of geometrically finite metrics uniformizing M and $\lambda \in \mathcal{D}(M)$ is a measured geodesic lamination such that $l_{\rho_n}(\lambda)$ is bounded, then the sequence (ρ_n) contains an algebraically converging subsequence.

We will also discuss the action of the modular group on $\mathcal{D}(M)$.

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2 Definitions

2.1 Geodesic Laminations

Let S be a closed surface endowed with a complete hyperbolic metric; a *geodesic lamination* on S is a compact subset that is the disjoint union of complete embedded geodesics. Using the fact that two complete hyperbolic metrics on S are quasi-isometric, this definition can be made independent of the chosen metric on S (see [Ot2] for example). A geodesic lamination whose leaves are all closed is called a *multi-curve*. If each half-leaf of a geodesic lamination L is dense in L, then L is *minimal*. Such a minimal geodesic lamination is either a simple closed curve or an *irrational lamination*. A leaf l of a geodesic lamination L is *recurrent* if it lies in a minimal geodesic lamination. Any geodesic lamination is the disjoint union of finitely many minimal laminations and non-recurrent leaves. A leaf is said to be an *isolated* leaf if it is either a non-recurrent leaf or a compact leaf without any leaf spiraling toward it.

Let L be a connected geodesic lamination which is not a simple closed curve and let us denote by $\bar{S}(L)$ the smallest surface with geodesic boundary containing L. Inside $\bar{S}(L)$ there are finitely many closed geodesics (including the components of $\partial\bar{S}(L)$) disjoint from L and these closed geodesics do not intersect each other (cf. [Le1]); let us denote by $\partial'\bar{S}(L) \supset \partial\bar{S}(L)$ the union of these geodesics. Let us remove from $\bar{S}(L)$ a small tubular neighbourhood of $\partial'\bar{S}(L)$ and let $S(L)$ be the resulting surface. We will call $S(L)$ the *surface embraced* by the geodesic lamination L and $\partial'\bar{S}(L)$ the *effective boundary* of $S(L)$. If L is a simple closed curve, let us define $S(L)$ to be an annular neighbourhood of L and $\partial'\bar{S}(L) = L$. If L is not connected, $S(L)$ is the disjoint union of the surfaces embraced by the connected components of L and $\partial'\bar{S}(L) = \bigcup_{\{L^i \text{ is a component of } L\}} \partial'\bar{S}(L^i)$.

A *measured geodesic lamination* λ is a transverse measure for some geodesic lamination $|\lambda|$: any arc $k \approx [0,1]$ embedded in S transversely to $|\lambda|$, such that

$\partial k \subset S - \lambda$, is endowed with an additive measure $d\lambda$ such that :

- the support of $d\lambda_{|k}$ is $|\lambda| \cap k$;

- if an arc k can be homotoped into k' by a homotopy respecting $|\lambda|$ then $\int_k d\lambda = \int_{k'} d\lambda$.

We will denote by $\mathcal{ML}(S)$ the space of measured geodesic lamination topologised with the topology of weak* convergence. We will denote by $|\lambda|$ the support of a measured geodesic lamination λ.

Let γ be a weighted simple closed geodesic with support $|\gamma|$ and weight w and let λ be a measured geodesic lamination, the intersection number between γ and λ is defined by $i(\gamma, \lambda) = w \int_{|\gamma|} d\lambda$. The weighted simple closed curves are dense in $\mathcal{ML}(S)$ and this intersection number extends continuously to a function $i : \mathcal{ML}(S) \times \mathcal{ML}(S) \to \mathbb{R}$ (cf. [Bo1]). A measured geodesic lamination λ is *arational* if for any simple closed curve $i(c, \lambda) = \int_c d\lambda > 0$.

2.2 Real trees

An $\mathbb{R}$-tree $\mathcal{T}$ is a metric space such any two points x, y can be joined by a unique simple arc. Let G be a group acting by isometries on an $\mathbb{R}$-tree $\mathcal{T}$; the action is *minimal* if there is no proper invariant subtree and *small* if the stabilizer of any non-degenerate arc is virtually abelian.

A G-equivariant map ϕ between two $\mathbb{R}$-trees $\mathcal{T}$ and $\mathcal{T}'$ is a *morphism* if and only if every point $p \in \mathcal{T}$ lies in a non-degenerate segment $[a, b]$ (but p may be a vertex of $[a, b]$) such that the restriction $\phi_{|[a,b]}$ is an isometry. The point p is a *branching point* if there is no segment $[a, b]$ such that $\phi_{|[a,b]}$ is an isometry and that $p \in]a, b[$.

Let S be a connected hyperbolic surface and let $q : \mathbb{H}^2 \to S$ be the covering projection. Let $L \subset S$ be a geodesic lamination and let $\pi_1(S) \curvearrowright \mathcal{T}$ be a minimal action of $\pi_1(S)$ on an $\mathbb{R}$-tree $\mathcal{T}$; L is realized in $\mathcal{T}$ if there is a continuous equivariant map $\mathbb{H}^2 \to \mathcal{T}$ whose restriction to any lift of a leaf of L is injective.

Let $\lambda \in \mathcal{ML}(S)$ be a measured geodesic lamination; following [MoO], we will define the dual tree of λ. Consider the following metric space $pre\mathcal{T}_\lambda$: the points of $pre\mathcal{T}_\lambda$ are the complementary regions of $q^{-1}(\lambda)$ in $\mathbb{H}^2$, where $q : \mathbb{H}^2 \to S$ is the covering projection and the distance $d : \mathcal{T}_\lambda \times \mathcal{T}_\lambda \to \mathbb{R}$ is defined as follows. Let R_0 and R_1 be two complementary regions and choose a geodesic segment $k \subset \mathbb{H}^2$ whose vertices lie in R_0 and R_1; we set $d(R_0, R_1)$ to be the $q^{-1}(\lambda)$-measure of k. Then, there is a unique (up to isometry) $\mathbb{R}$-tree $\mathcal{T}_\lambda$ and an isometric embedding $e : pre\mathcal{T}_\lambda \to \mathcal{T}_\lambda$ such that **any point** of $\mathcal{T}_\lambda$ lies in a segment with endpoints in $e(pre\mathcal{T}_\lambda)$ (cf. [GiS]). The covering transformations yield an isometric action of $\pi_1(M)$ on $\mathcal{T}_\lambda$; if $\delta_\lambda(c)$ is the distance of translation of an isometry of $\mathcal{T}_\lambda$ corresponding to a simple closed curve c, we have $\delta_\lambda(c) = i(c, \lambda)$. This construction yields a natural projection $\mathbb{H}^2 - q^{-1}(\lambda) \to \mathcal{T}_\lambda$. If λ does not have closed leaves, this projection extends continuously to a map $\pi_\lambda : \mathbb{H}^2 \to \mathcal{T}_\lambda$. Otherwise, replacing closed leaves of λ by foliated annuli endowed with uniform transverse measures, we get also a continuous map $\pi_\lambda : \mathbb{H}^2 \to \mathcal{T}_\lambda$ (cf. [Ot2]).

2.3 Train tracks

A *train track* τ in S is the union of finitely many "rectangles" b_i called the *branches* and satisfying:

- any branch b_i is an imbedded rectangle $[0, 1] \times [0, 1]$ such that the preimage of the double points is a segment of $\{0\} \times [0, 1]$ and a segment of $\{1\} \times [0, 1]$;

- the intersection of two different branches is either empty or a non-degenerate segment lying in the vertical sides $\{0\} \times [0,1]$ and $\{1\} \times [0,1]$;

- any connected component of the union of the vertical sides is a simple arc embedded in $\partial_{\chi<0} M$.

A connected component of the union of the vertical sides is a *switch*. In each branch the segments $\{p\} \times [0,1]$ are the *ties* and the segments $[0,1] \times \{p\}$ are the *rails*.

A geodesic lamination L is *carried by a train track* τ when:

- L lies in τ;

- for each branch b_i of τ, $L \cap b_i$ is not empty, lies in the image of $[0,1] \times]0,1[$ and each leaf of L is transverse to the ties.

Notice that, in some papers, a geodesic lamination satisfying the above is said to be "minimally carried" by τ.

A measured geodesic lamination λ is carried by a train track τ if its support $|\lambda|$ is carried by τ.

Let S be a hyperbolic surface, let $\tau \subset S$ be a train track and let $\pi_1(M) \curvearrowright \mathcal{T}$ be a minimal action of $\pi_1(M)$ on an $\mathbb{R}$-tree $\mathcal{T}$. Let $q^{-1}(\tau) \subset \mathbb{H}^2$ be the preimage of τ under the covering projection; a *weak realization* of τ in $\mathcal{T}$, is a $\pi_1(M)$-equivariant continuous map $\pi : q^{-1}(\tau) \to \mathcal{T}$ such that π is constant on the ties of $q^{-1}(\tau)$, monotone and not constant on the rails and that the images of two adjacents branches lying on opposite sides of the same switch have disjoint interiors.

2.4 3-manifolds

Let M be a 3-manifold, M is *irreducible* if any sphere embedded in M bounds a ball. We will say that M is a *hyperbolic manifold* if its interior can be endowed with a complete hyperbolic metric. Let Σ be a subsurface of ∂M; an *essential disc* in (M, Σ) is a disc D properly embedded in (M, Σ) that can not be mapped to ∂M by a homotopy fixing ∂D. The simple closed curve ∂D is a *meridian* curve. The manifold M is *boundary irreducible* if there is no essential disc in $(M, \partial M)$. An *essential annulus* in (M, Σ) is an incompressible annulus A properly embedded in (M, Σ) which can not be mapped to ∂M by a homotopy fixing ∂A. Let A be an essential annulus in M; if one component of ∂A lies in a toric component of ∂M we will call the other component of ∂A a *parabolic curve*.

Let $m \subset \partial M$ be a simple closed curve; a simple arc k such that $k \cap m = \partial k$ is an m-wave if there is an arc $k' \subset m$ such that $k' \cup k$ bounds an essential disc. A leaf $\tilde{l}$ of a geodesic lamination $\tilde{L} \subset \partial \tilde{M}$ is *homoclinic* if it contains two sequences of points (x_n) and (y_n) such that the distance between the points x_n and y_n measured on $\tilde{l}$ goes to ∞ whereas their distance measured in $\tilde{M}$ is bounded. A leaf l of a geodesic lamination $L \subset \partial M$ is *homoclinic* if a (any) lift of l to $\partial \tilde{M}$ is a homoclinic leaf. Notice that, with this definition, a meridian or a leaf spiralling around a meridian is homoclinic.

Let $\rho : \pi_1(M) \to Isom(\mathbb{H}^3)$ be a faithful discrete representation such that $\mathbb{H}^3/\rho(\pi_1(M))$ is homeomorphic to the interior of M. Let $L_\rho \subset S^2 = \partial\overline{\mathbb{H}}^3$ be the limit set of $\rho(\pi_1(M))$, let $C(\rho) \subset \mathbb{H}^3$ be the convex hull of L_ρ and let $C(\rho)^{ep}$ be the intersection of $C(\rho)$ with the preimage of the thick part of $\mathbb{H}^3/\rho(\pi_1(M))$. The quotient $N(\rho)$ of $C(\rho)$ by

$\rho(\pi_1(M))$ is the convex core of ρ and ρ is said to be *geometrically finite* if $N(\rho)$ has finite volume. A geometrically finite representation $\rho : \pi_1(M) \to Isom(\mathbb{H}^3)$ such that $\mathbb{H}^3/\rho(\pi_1(M))$ is homeomorphic to the interior of M is said to *uniformize* M. If ρ uniformize M, there is a natural homeomorphism (defined up to homotopy) $h : \tilde{M} \to C(\rho)^{ep}$ coming from the retraction map $S^2 - L_\rho \to C(\rho)^{ep}$. Let us choose a geometrically finite representation ρ with only rank 2 maximal parabolic subgroups (namely the maximal subgroups of $\rho(\pi_1(M))$ containing only parabolic isometries have rank 2). We will define the compactification $\overline{\tilde{M}}$ of $\tilde{M}$ as the closure of $h(\tilde{M}) = C(\rho)^{ep}$ in the usual unit ball compactification of $\mathbb{H}^3$. This compactification does not depend on the choice of the representation ρ (see [Le1, section 2.1]). We will call this compactification the Floyd-Gromov compactification of $\tilde{M}$.

Let $\tilde{l}_+ \subset \partial\tilde{M}$ be a half-geodesic and let $\bar{\tilde{l}}_+$ be its closure in $\overline{\tilde{M}}$; we will say that $\tilde{l}_+$ *has a well defined endpoint* if $\bar{\tilde{l}}_+ - \tilde{l}_+$ contains one point. We will say that a geodesic $\tilde{l} \subset \partial\tilde{M}$ has two well defined endpoints if $\tilde{l}$ contains two disjoints half geodesics each having a well defined endpoint. Two distincts leaves $\tilde{l}_1$ and $\tilde{l}_2$ of a geodesic lamination $\tilde{L} \subset \partial\tilde{M}$ will be said to be *biasymptotic* if they both have two well defined endpoints in $\overline{\tilde{M}}$ and if the endpoints of $\tilde{l}_1$ are the same as the endpoints of $\tilde{l}_2$. A geodesic lamination $A \subset \partial M$ is *annular* if the preimage of A in $\partial\tilde{M}$ contains a pair of biasymptotic leaves.

2.5 Pleated surfaces

Let $\rho : \pi_1(M) \to Isom(\mathbb{H}^3)$ be a discrete faithful representation and let $N = \mathbb{H}^3/\rho(\pi_1(M)$. A *pleated surface* in N is a map $f : S \to N$ from a surface S to N with the following properties :

- the path metric obtained by pulling back the hyperbolic metric of N by f is a hyperbolic metric s on S;

- every point in S liess in the interior of some s-geodesic arc that is mapped to a geodesic arc in N;

The *pleating locus* of a pleated surface is the set of points of S where the map fails to be a local isometry. The pleating locus of a pleated map is a geodesic lamination (cf. [Th1]).

Let $\rho : \pi_1(M) \to Isom(\mathbb{H}^3)$ be a discrete faithful representation such that there is a homeomorphism $h : int(M) \to N = \mathbb{H}^3/\rho(\pi_1(M))$ and let $S \subset M$ be a properly embedded surface homeomorphic and homotopic to ∂M. A measured geodesic lamination $\lambda \in \mathcal{ML}(\partial M)$ is realized by a pleated surface in N if there is a pleated surface $f : S \to N$ homotopic to $h_{|S}$ such that the restriction of f to the support of λ is an isometry.

2.6 Masur domain

Let M be a compression body; its boundary has a unique compressible component, the *exterior boundary* that we will denote by $\partial_e M$. Let $\mathcal{PML}(\partial_e M)$ be the space of projective measured geodesic laminations on $\partial_e M$ and let $\mathcal{M}'$ be the closure in $\mathcal{PML}(\partial_e M)$ of the set of projective classes of weighted meridians. The compression body M is said to be a *small compression body* if it is the connected sum along the boundary of two I-bundles over closed surfaces or of a solid torus and of an I-bundle over a closed surface and is said to be a *large compression body* otherwise. When M is a large compression body, the Masur domain is defined as follows :

$$\mathcal{O} = \{\lambda \in \mathcal{PML}(\partial_e M) | i(\lambda, \mu) > 0 \text{ for any } \mu \in \mathcal{M}'\}.$$

When M is a small compression body, the definition is the following one

$\mathcal{O} = \{\lambda \in \mathcal{PML}(\partial_e M) | \, i(\lambda, \nu) > 0$ for any $\nu \in \mathcal{PML}(\partial_e BC)$ such that there is $\mu \in \mathcal{M}'$ with $i(\mu, \nu) = 0\}$.

We will denote by $\hat{\mathcal{O}} \subset \mathcal{ML}(\partial M)$ the set of measured geodesic laminations whose projective class lies in $\mathcal{O}$.

Let M be an orientable hyperbolic 3-manifold such that ∂M has negative Euler characteristic. We will say that a measured geodesic lamination $\lambda \in \mathcal{ML}(\partial M)$ is *doubly incompressible* if and only if :

- $\exists \eta > 0$ such that $i(\lambda, \partial E) \geq \eta$ for any essential annulus or disc E.

We will denote by $\mathcal{D}(M) \subset \mathcal{ML}(\partial M)$ the set of doubly incompressible measured geodesic laminations.

Doubly incompressible multi-curve were first introduced by W. Thurston in [Th4] and we have the following equivalence : $(\partial M, |\gamma|, \subset)$ is doubly incompressible (in the sense of [Th4]) if and only if there is a weighted multi-curve $\gamma \subset \mathcal{ML}(\partial M)$ with support $|\gamma|$ satisfying the condition above except in the following situation (in which γ lies in $\mathcal{D}(M)$ but $(\partial M, |\gamma|, \subset)$ is not doubly incompressible in Thurston's sense):

- $(-)$ there is a homeomorphism between M and an I-bundle over a pair of pants P such that $|\gamma|$ is mapped to a section of the bundle over ∂P.

The set $\mathcal{D}(M)$ of doubly incompressible measured geodesic laminations is the extension of Masur domain we will study in this paper.

3 Relations between $\mathcal{O}(M)$, $\mathcal{D}(M)$ and $\mathcal{P}(M)$

When a statement deals with the Masur domain, it means that we have assumed that M is a compression body.

Lemma 3.1. *The set $\hat{\mathcal{O}}$ is a subset of $\mathcal{D}(M)$.*

Proof. Let $\lambda \notin \mathcal{D}(M)$ be a measured geodesic lamination. We will show, using the following lemma of [Ot1], that $\lambda \notin \hat{\mathcal{O}}$.

Lemma 3.2 ([Ot1])**.** *Let E be an essential annulus in a large compression body M; then there is a projective measured geodesic lamination $\mu \in \mathcal{M}'$ with support lying in ∂E.*

Proof. Since [Ot1] is not published, we will write the details of the proof. The boundary of ∂M has only one compressible component $\partial_e M$ called the exterior boundary. Let us choose a complete hyperbolic metric on $\partial_e M$.

Claim 3.3. *Let $c \subset \partial_e M$ be a simple closed curve that is disjoint from one non separating meridian or from two separating meridians; then there is a projective measured geodesic lamination $\mu \in \mathcal{M}'$ whose support is c.*

Proof. Let us first consider that there is a non separating meridian m disjoint from c. Let D be an essential disc bounded by c. Since c does not separate ∂M, there is a sequence of simple closed curves (c_i) that approximates c and intersect m in one point, namely the sequence (c_i) converges to c in $\mathcal{PML}(\partial M)$. Consider a small neighbourhood $\mathcal{V}_i$ of $D \cup c_i$ in M. The closure of $\partial \mathcal{V}_i - \partial M$ is an essential disc D_i and the sequence (∂D_i) converges to c in $\mathcal{PML}(\partial M)$.

Let us now assume that there are two disjoint separating meridians m_1 and m_2 which do not intersect c. Let D_1 and D_2 be two essential discs bounded by m_1 and m_2 respectively. Let N be the closure of the connected component of $M-(D_1 \cup D_2)$ whose boundary contains c. If N intersects D_1 and D_2, we can approximate c by a sequence of arcs k_i joining m_1 to m_2. Let $\mathcal{V}_i$ be a small neighbourhood of $D_1 \cup k \cup D_2$. The closure of $\partial\mathcal{V}_i - \partial M$ is an essential disc Δ_i and the sequence $(\partial \Delta_i)$ converges to c in $\mathcal{PML}(\partial M)$.

If N intersects only one disc D_1 or D_2, by considering an arc in $\partial M - N$ joining D_1 and D_2, we can construct an essential disc D_3 such that one component of $M-(D_1 \cup D_3)$ or of $M-(D_2 \cup D_3)$ contains c and intersects D_1 and D_3 or D_2 and D_3. Thus we are in the previous case and can conclude as above. □

To prove Lemma 3.2, it remains to consider the case where there is at most one meridian disjoint from E and this meridian separates M.

Let us assume that the two components of ∂E are not homotopic in ∂M. Since M is a large compression body, E intersects a meridian c. Let us choose an orientation for E and let $\psi : M \to M$ be the Dehn twist along E. The curve $\psi^n(c)$ is a meridian. The restriction of ψ_n to ∂M is a Dehn twist along ∂E. It follows that the sequence $(\psi^n(c))$ tends to a projective measured geodesic lamination $\mu \in \mathcal{M}'$ with $|\mu| \subset \partial E$.

Consider now that there is an annulus $E'\partial M$ with $\partial E' = \partial E$. By cutting M along an essential disc disjoint from E (if there is one, we can assume that E intersects any essential disc in M. Since M is atoroidal and $E \cup E'$ bounds a solid torus $T \subset M$. Furthermore each component of $\partial E'$ represent an element in $\pi_1(M)$ which is divisible. It follows that when M is described as the connected sum along the boundary of tori and I-bundle over closed surfaces, T does not go through an I-bundle over a closed surface. Since T intersects any essential disc, we get that M is a solid torus. Recalling that we may have cut M along an essential disc, we conclude that M was originally the connected sum along the boundary of a solid torus and an I-bundle over a closed surface. This contradicts our asumption that M is a large compression body.

□

Remark. If E is an essential annulus in a small compression body, either ∂E intersects a meridian and from the above a measured geodesic sublamination of ∂E lies in $\mathcal{M}'$, or ∂E is disjoint from the meridian. ◇

Let λ be a measured geodesic lamination such that $\lambda \notin \mathcal{D}(M)$. Then there is a sequence of essential discs or annuli $E_n \subset M$ such that $i(\lambda, \partial E_n) \longrightarrow 0$. We will show that $\lambda \notin \hat{\mathcal{O}}$.

We will first assume that M is a large compression body. By Lemma 3.2, there is a sequence of multi-curves (e_n) such that $e_n \subset \partial E_n$ and that $e_n \in \mathcal{M}'$. Let $\varepsilon > 0$ and let εe_n be the weighted multi-curve obtained by endowing each leaf of e_n with a Dirac mass with weight ε. Up to extracting a subsequence, there is a sequence (ε_n) converging to 0 such that the sequence $(\varepsilon_n e_n)$ converges to some measured geodesic lamination α. Since $\varepsilon_n e_n \in \mathcal{M}'$ for any n, then $\alpha \in \mathcal{M}'$. Since $\varepsilon_n \longrightarrow 0$, we have $i(\lambda, \alpha) = 0$ hence $\lambda \notin \hat{\mathcal{O}}$.

Let us now assume that M is a small compression body. By the proof of Lemma 3.2, for each n, either E_n is disjoint from an essential meridian or a connected component of ∂E_n is the support of an element of $\mathcal{M}'$. Especially, for any n, there is $\mu_n \in \mathcal{M}'$ with $i(\mu_n, e_n) = 0$. Furthermore, we can choose the μ_n such that a subsequence of (μ_n) converges in $\mathcal{ML}(\partial M)$ to a measured geodesic lamination $\mu \in \mathcal{M}'$. We get then $i(\alpha, \mu) = 0$ and $i(\alpha, \lambda) = 0$ hence $\lambda \notin \hat{\mathcal{O}}$. Thus we have shown that if $\lambda \notin \mathcal{D}(M)$, then $\lambda \notin \hat{\mathcal{O}}$. □

The opposite is not true but we have the following :

Lemma 3.4. *Let $\lambda \in \mathcal{D}(M)$ be an arational measured geodesic lamination; then λ lies in $\hat{\mathcal{O}}$.*

Proof. Let us assume the contrary; if M is a large compression body, there is $\mu \in \mathcal{M}'$ such that $i(\mu, \lambda) = 0$. It follows from the assumption that λ is arational that λ and μ share the same support $|\mu|$. Since $\mu \in \mathcal{M}'$, there is a sequence of meridians $c_n \subset \partial M$ and a sequence $\varepsilon_n \longrightarrow 0$ such that $\varepsilon_n c_n$ converges to μ in the topology of $\mathcal{ML}(\partial M)$. Up to extracting a subsequence, (c_n) converges in the Hausdorff topology to a geodesic lamination L and we have $|\mu| \subset L$. By Casson's criterion (cf. [Ot1], [Le1, Theorem B.1] or [Le2]), L contains a homoclinic leaf l. Since $|\mu| \subset L$ is the support of λ, l does not intersect λ transversely. This contradicts Lemma 3.6 below.

If M is a small compression body, ∂M contains a unique meridian c. Let us assume that $\lambda \notin \hat{\mathcal{O}}$; then there is $\mu \in \mathcal{ML}(\partial_e M)$ such that $i(c, \mu) = 0$ and $i(\lambda, \mu) = 0$. Since λ is arational and $i(\lambda, \mu) = 0$, μ is also arational. This contradicts the fact that $i(c, \mu) = 0$. □

In [Le1] (see also [Le2]), one studied the subset $\mathcal{P}(M)$ of $\mathcal{ML}(\partial M)$ defined as follows. Let $\lambda \in \mathcal{ML}(\partial M)$ be a measured geodesic lamination; then $\lambda \in \mathcal{P}(M)$ if and only if :

- *a*) no closed leaf of λ has a weight greater than π;

- *b*) $\exists \eta > 0$ such that, for any essential annulus E, $i(\partial E, \lambda) \geq \eta$;

- *c*) $i(\lambda, \partial D) > 2\pi$ for any essential disc D.

Let $\rho : \pi_1(M) \to Isom(\mathbb{H}^3)$ be a geometrically finite representation uniformizing M and let h be an isotopy class of homeomorphisms $M \to N(\rho)^{cp}$ homotopic to the identity; we will denote by $\mathcal{GF}(M)$ the set of such pairs (ρ, h). There is a well defined map $b : \mathcal{GF}(M) \to \mathcal{ML}(\partial M)$ which to a pair (ρ, h) associates the preimage under h of the bending measured geodesic lamination of $N(\rho)$, let us call this map the bending map. It is shown in [BoO] and [Le1] that $\mathcal{P}(M)$ is the image of b.

In [Le1], it was proved that a measured geodesic lamination lying in $\mathcal{P}(M)$ intersects transversely all the homoclinic leaves and all the annular laminations. In order to get the same property for the laminations lying in $\mathcal{D}(M)$, we will discuss the relationships between $\mathcal{P}(M)$ and $\mathcal{D}(M)$.

We clearly have $\mathcal{P}(\partial M) \subset \mathcal{D}(\mathcal{M})$, conversely, we have :

Lemma 3.5. *Let $\lambda \in \mathcal{D}(\mathcal{M})$ be a measured geodesic lamination not satisfying the condition $(-)$, then there is a measured geodesic lamination $\alpha \in \mathcal{P}(M)$ with the same support as λ.*

Proof. Since $\lambda \in \mathcal{D}(\mathcal{M})$, $\exists \eta > 0$ such that $i(\partial E, \lambda) > \eta$ for any essential annulus or disc E. Let $\frac{2\pi}{\eta}\lambda$ be the measured geodesic lamination obtained by multiplying the measure λ by $\frac{2\pi}{\eta}$; then $\frac{2\pi}{\eta}\lambda$ satisfies the properties *b*) and *c*) above. Let $\lambda^{(p)}$ be the union of the leaves of $\frac{2\pi}{\eta}\lambda$

with a weight greater than π and let α be the measured geodesic lamination obtained from $\frac{2\pi}{\eta}\lambda$ by decreasing the weight of the leaves of $\lambda^{(p)}$ to π. This measured geodesic lamination α satisfies a) and b), let us show that it satisfies also c).

Let $D \subset M$ be an essential disc; then $i(\frac{2\pi}{\eta}\lambda, \partial D) > 2\pi$. If ∂D does not intersect $\lambda^{(p)}$ transversely, then $i(\alpha, \partial D) = i(\frac{2\pi}{\eta}\lambda, \partial D) > 2\pi$.

If ∂D intersects $\lambda^{(p)}$ in one point x, let c be the leaf of $\lambda^{(p)}$ containing x. Let $\mathcal{V}$ be a small neighbourhood of $c \cup D$; $\mathcal{V}$ is a solid torus. Let D' be the closure of $\partial\mathcal{V} - \partial M$, D' is a disc properly embedded in M which does not intersect $\lambda^{(p)}$. Hence we have $i(\partial D', \alpha) = i(\partial D', \frac{2\pi}{\eta}\lambda)$. If D' is not an essential disc, then $\partial D'$ bounds a disc $D'' \subset \partial M$. Since M is irreducible, $D' \cup D''$ bounds a ball $B \subset M$ and $M = B \cup \mathcal{V}$ is a solid torus. By assumption, M is not a solid torus hence D' is an essential disc and $i(\partial D', \frac{2\pi}{\eta}\lambda) > 2\pi$. Since $i(\partial D', \alpha) \leq 2(i(\partial D, \alpha) - \pi)$, we have $i(\partial D, \alpha) \geq \frac{i(\partial D', \alpha)}{2} + \pi = \frac{i(\partial D', \frac{2\pi}{\eta}\lambda)}{2} + \pi > 2\pi$.

If ∂D intersects $\lambda^{(p)}$ in two points x and y, we have $i(\alpha, \partial D) = 2\pi + i(\frac{2\pi}{\eta}\lambda - \lambda^{(p)}, \partial D)$. Hence we just have to show that $i(\lambda - \lambda^{(p)}, \partial D) > 0$. Assuming the contrary, we have $\lambda \cap \partial D = \{x, y\}$. If x and y lie in two distincts leaves $c \subset |\lambda|$ and $d \subset |\lambda|$, let $\mathcal{V}$ be a small neighbourhood of $c \cup d \cup D$; $\mathcal{V}$ is an I-bundle over a pair of pants. The closure of $\partial\mathcal{V} - \partial M$ is an annulus with boundary not intersecting $|\lambda|$. By condition b), this annulus is not essential. It follows that M is an I-bundle over a pair of pants P and that $|\lambda|$ lies in a section of the bundle over ∂P. This contradicts our assumptions hence x and y lie in the same leaf c of $\lambda^{(p)}$.

Let $\mathcal{V}$ be a small neighbourhood of $c \cup D$; it is again an I-bundle over a pair of pants. If the tangents vectors $\frac{dc}{dt}_{|x}$ and $\frac{dc}{dt}_{|y}$ do not point to the same side of ∂D, the closure of $\partial\mathcal{V} - \partial M$ is the union of two annuli with boundaries not intersecting λ. This yields the same contradiction as above.

Next let us consider the case where $\frac{dc}{dt}_{|x}$ and $\frac{dc}{dt}_{|y}$ point to the same side of ∂D. Let k be a connected component of $c - \{x, y\}$ and let $\mathcal{V}'$ be a small neighbourhood of $k \cup D$; the closure of $\partial\mathcal{V}' - \partial M$ is an essential disc D'. Replacing D by D', we are in the situation of the previous paragraph and get the same contradiction.

If ∂D and $\lambda^{(p)}$ intersect each other in more than 2 points, $i(\lambda', \partial D) \geq 3\pi$. □

Combining Lemma 3.5 and results of [Le1] (see also [Le2]) we get the following :

Lemma 3.6. *A measured geodesic lamination $\lambda \in \mathcal{D}(M)$ not satisfying the condition $(-)$ has the following property :*

- *λ intersects transversely any annular lamination and any geodesic lamination containing a homoclinic leaf.*

Remark. Let us add a few comments about the case where λ satisfies the condition $(-)$. Any homoclinic leaf l intersects λ at least once. If an annular geodesic lamination A does not intersect λ transversely, then A contains two disjoint half-leaves both spiraling in the same direction toward the same leaf of λ. This can not happen for a Hausdorff limit of multi-curves. Therefore λ has the property above if we consider only annular laminations that are Hausdorff limits of multi-curves. ◇

4 Topological properties of $\mathcal{D}(M)$

Lemma 4.1. *The set $\mathcal{D}(M)$ is an open set.*

Proof. Let us assume the contrary. Then there are $\lambda \in \mathcal{D}(M)$ and a sequence of measured geodesic laminations $\lambda_n \notin \mathcal{D}(M)$ converging to λ. Therefore there is a sequence of essential discs or annuli E_n such that $i(\lambda_n, \partial E_n) \longrightarrow 0$. Let us extract a subsequence such that ∂E_n converge in the Hausdorff topology to a geodesic lamination A. Then A does not intersect λ transversely. By [Le1] (see also [Le2]) either A contains a homoclinic leaf ([Le1, Theorem B1]) or A is annular ([Le1, Lemma C2]), both contradicting Lemma 3.6. □

A train track τ carrying a measured geodesic lamination is *complete* if it is not a subtrack of a train track carrying a measured geodesic lamination (cf. [Pe]).

Any measured geodesic lamination λ is carried by some (maybe many) complete train track τ. The weight system on a complete train track gives rise to a coordinate system for a simplex of the piecewise linear manifold $\mathcal{ML}(\partial M)$. The *rational depth* of a measured geodesic lamination λ is the dimension of the rational vector space of linear functions with rational coefficients (from the simplex previously defined to $\mathbb{R}$) vanishing on the coordinates of λ. Let us denote by $\mathcal{I}(\partial M)$ the set of measured geodesic laminations with rational depth equal to 0 or 1. If a measured geodesic lamination λ lies in $\mathcal{I}$, either λ is arational or there is a closed leaf c of λ such that λ is arational in $\partial M - c$ (cf. [Th1, Proposition 9.5.12]). By Lemma 3.5 and [Le3, Lemma 2.5], the proof of Lemma 3.4 holds also in the second case, namely if $\lambda \in \mathcal{D}(M)$ and if there is closed leaf c of λ such that λ is arational in $\partial M - c$, then $\lambda \in \hat{\mathcal{O}}$. The set $\mathcal{I}$ is a dense open subset of $\mathcal{ML}(\partial M)$ (cf. [Th1, chap 9]).

Proposition 4.2. *The sets $\mathcal{D}(M)$ and $\hat{\mathcal{O}}$ are pathwise connected.*

Proof. Let $\lambda_1, \lambda_2 \in \hat{\mathcal{O}}$. By [Ma1], the arational measured geodesic laminations are dense in $\mathcal{ML}(\partial M)$. Since $\hat{\mathcal{O}}$ is open, there are two arational measured geodesic laminations α_1 and $\alpha_2 \in \hat{\mathcal{O}}$ such that λ_j is connected to α_j by a path $k_j \subset \hat{\mathcal{O}}$.

Since $\alpha_j \in \hat{\mathcal{O}} \subset \mathcal{D}(M)$ there is $\eta > 0$ such that $i(\alpha_j, \partial E) > \eta$ for any essential disc or annulus $E \subset M$. Since α_i is arational, it has no closed leaf and, by the proof of Lemma 3.5, we have $\frac{2\pi}{\eta}\alpha_j \in \mathcal{P}(M)$. Let $\mathcal{CC}(M) \subset \mathcal{GF}(M)$ be the set of hyperbolic metrics uniformizing M and having only rank 2 cusps. By results of Ahlfors-Bers ([Ber]), $\mathcal{CC}(M)$ is homeomorphic to the cartesian product of the Teichmüller spaces of the connected components of $\partial_{\chi<0}M$, indeed $\mathcal{CC}(M)$ is pathwise connected. Let $\mathcal{P}_{nc}(M)$ be the set of measured geodesic laminations lying in $\mathcal{P}(M)$ and having no closed leaves with weight π. By [Le1] (see also [Le2]) $\mathcal{P}_{nc}(M)$ is the image of $\mathcal{CC}(M)$ by the bending map. By [KeS] and [Bo2], the bending map is continuous on $\mathcal{CC}(M)$ hence $\mathcal{P}_{nc}(M)$ is pathwise connected. Since $\frac{2\pi}{\eta}\alpha_j$ has no closed leaf, $\frac{2\pi}{\eta}\alpha_j \in \mathcal{P}_{nc}(M)$, therefore there is a path $\alpha : [0,1] \to \mathcal{P}(M)$ such that $\alpha(0) = \frac{2\pi}{\eta}\alpha_1$ and that $\alpha(1) = \frac{2\pi}{\eta}\alpha_2$. Since $\mathcal{D}(M)$ is open, we can change α so that we have $\alpha(t) \in \mathcal{I} \cap \mathcal{D}(M)$ for any $t \in [0,1]$ (cf. [Th1]). Thus $\alpha(t)$ is an arational lamination (up to cutting ∂M along a closed leave of $\alpha(t)$ if there is one) lying in $\mathcal{D}(M)$. From Lemma 3.4 we get $\alpha(t) \in \hat{\mathcal{O}}$ for any $t \in [0,1]$.

Let $\kappa_j : [0,1] \to \hat{\mathcal{O}}$ be the path $\kappa_j(t) = (1 - t + t\frac{2\pi}{\eta})\alpha_j$. The union of the paths k_j, κ_j for $j = 1, 2$ and of the path $\alpha([0,1])$ is a path lying in $\hat{\mathcal{O}}$ joining λ_1 to λ_2.

We have proved that $\hat{\mathcal{O}}$ is pathwise connected. Taking $\lambda_1, \lambda_2 \in \mathcal{D}(M)$ at the beginning of this proof, we get that $\mathcal{D}(M)$ is also pathwise connected. □

5 Pleated surfaces

Theorem 5.1. *Let M be an orientable 3-manifold, let $\rho : \pi_1(M) \to Isom(\mathbb{H}^3)$ be a geometrically finite representation uniformizing N and having only rank 2 maximal parabolic* *subgroups and let $h : N = \mathbb{H}^3/\rho(\pi_1(M)) \to int(M)$ be a homeomorphism; then any measured geodesic lamination $\lambda \in \mathcal{D}(M)$ is realized by a pleated surface in N.*

Proof. If M is a compression body and λ is arational, then λ lies in the Masur domain and the theorem has been proved by Otal ([Ot1]). If M is boundary irreducible, then any geodesic lamination is realized in N (see [CEG, chap. 5]). In order to prove our general statement, we will follow the main lines of Otal's proof.

Lemma 5.2. *Let $\lambda \in \mathcal{D}(M)$ be a weighted multi-curve, then λ is realized by a pleated surface in N.*

Proof. Let us extend $|\lambda|$ to a geodesic lamination L (namely $|\lambda| \subset L$) such that all the components of $\partial M - L$ are triangles and that L has finitely many leaves. Since $\lambda \in \mathcal{D}(M)$ and since ρ has only rank 2 cusps, any closed leaf of L is homotopic to a closed geodesic in N. Let $S \subset M$ be a properly embedded surface homeomorphic and homotopic to ∂M and let us change the restriction of h to S by a homotopy in order to get a map $f : S \to N$ mapping the closed leaves of L into closed geodesics. For each connected component of S, let us lift this to a map $\hat{f} : \mathbb{H}^2 \to \mathbb{H}^3$; this map $\hat{f}$ defines a map from the endpoints of the lifts of the leaves of L to L_ρ. Furthermore, if $\hat{l} \in \mathbb{H}^2$ is a lift of a leaf of L, by Lemma 3.6, the images of its two endpoints are distincts. Following [CEG, Theorem 5.3.6], this allows us to construct a pleated surface realizing L. $\square$

Now let us consider the general case. Let $\lambda \in \mathcal{D}(M)$ be a measured geodesic lamination; let λ_n be a sequence of weighted multi-curves such that $\lambda_n \longrightarrow \lambda$ in $\mathcal{ML}(\partial M)$ and that $|\lambda_n| \to |\lambda|$ in the Hausdorff topology. Since $\mathcal{D}(M)$ is open, $\lambda_n \in \mathcal{D}(M)$ for large n. Let γ be a weighted multi-curve with a maximal number of leaves such that $i(\lambda, \gamma) = 0$; since $\lambda_n \in \mathcal{D}(M)$ for large n, $\lambda_n \cup \gamma$ is also a measured geodesic lamination lying in $\mathcal{D}(M)$. By the previous lemma, $\lambda_n \cup \gamma$ is realized by a pleated surface $f_n : S \to N$. We will show that a subsequence of (f_n) converges to a pleated surfaces realizing λ.

Let us denote by s_n the metric on S induced by the map $f_n : S \to N$ and let us show that (s_n) contains a converging subsequence. First we will prove that the sequence of metrics (s_n) is bounded in the modular space. By Mumford's Lemma, it is sufficient to prove that the injectivity radius of s_n is bounded from below.

Claim 5.3. *Let (c_n) be a sequence of curves such that $l_{s_n}(c_n) \longrightarrow 0$ and let us extract a subsequence (c_n) which converges in the Hausdorff topology to a geodesic lamination C; then C does not intersects λ transversely.*

Proof. Let assume the contrary and let c be a leaf of C intersecting λ transversely. Since λ is recurrent, we can consider a segment $k = k([0,1])$ of $|\lambda|$ such that $k \cap C = \partial k$ and that $\frac{dk}{dt}(0)$ is close (for some reference metric on S) to $-\frac{dk}{dt}(1)$ and a short segment κ of c joining the ends of k so that we get a closed curve $d = k \cup \kappa$. Since $\lambda_n \longrightarrow \lambda$ and $c_n \longrightarrow C$, there exists arcs $k_n \subset \lambda_n$ and $\kappa_n \subset c_n$ near k and κ such that $d_n = k_n \cup \kappa_n$ is homotopic on S to d. Since $l_{s_n}(c_n) \longrightarrow 0$, c_n is the core of a very deep Margulis tube and $l_{s_n}(k_n) \longrightarrow \infty$. Since $l_{s_n}(\kappa_n) \leq l_{s_n}(c_n) \longrightarrow 0$ and $f_n(k_n) \subset f_n(\lambda_n)$ is a geodesic arc, $f_n(d_n) = f_n(k_n \cup \kappa_n)$ is a quasi-geodesic and is very close to the geodesic d_n^* of N in its homotopy class. This implies that $l_\rho(d_n^*) \longrightarrow \infty$ but d_n is homotopic to d so $d_n^* = d^*$ giving the expected contradiction. $\square$

Let (c_n) be a sequence of curves such that $l_{s_n}(c_n) \longrightarrow 0$. If we can extract a converging (in the Hausdorff topology) subsequence such that all the c_n are meridians then, by Casson's criterion (cf. [Ot1], [Le1, Theorem B.1]), the limit contains a homoclinic leaf. By Lemma 3.6

such a homoclinic leaf intersects λ transversely contradicting Claim 5.3. This implies that for large n, the c_n are not meridians. If we can extract a converging subsequence such that all the c_n are parabolic curves, then $i(c_n, \lambda) > \eta$ for any n, leading to the same contradiction.

It follows that, for large n, each $f_n(c_n)$ is homotopic to a closed geodesic c_n^* of N. But this would mean that $l_\rho(c_n^*) \longrightarrow 0$ and since N is geometrically finite, there is a uniform lower bound for the length of a closed geodesic. We get then from Mumford's Lemma ([CEG, Proposition 3.2.13]) :

Claim 5.4. *The sequence (s_n) is bounded in the moduli space.*

Let us now show that (s_n) is bounded in the Teichmüller space. By the previous claim, there exists a sequence (φ_n) of diffeomorphisms such that, up to extracting a subsequence, $(\varphi_n^* s_n)$ converges in the Teichmüller space to a metric s'_∞. By construction $l_{\varphi_n^* s_n}(\varphi_n^{-1}(\gamma)) = l_{s_n}(\gamma) = l_\rho(\gamma)$, therefore the s'_∞-length of the multi-curve $\varphi_n^{-1}(\gamma)$ is bounded. This implies that we can choose some n_0 and a subsequence such that any diffeomorphism $(\varphi_n^{-1} \circ \varphi_{n_0})$ preserves this multi-curve, component by component.

For large n, λ_n intersects transversely all the parabolic curves. Therefore λ_n lies in the thick part of N which is compact. It follows that all the $f_n(S)$ intersect the same compact subset of N. Using Ascoli's theorem we can choose a subsequence of (φ_n) such that the sequence of pleated surfaces $(f_n \circ \varphi_n)$ converges. This implies that the maps $f_n \circ \varphi_n$ are homotopic for n sufficiently large. Thus, up to changing n_0, the diffeomorphisms $\psi_n = \varphi_n^{-1} \circ \varphi_{n_0}$ are homotopic in M to the identity. Let R be a complementary region of γ. If the map $i^* : \pi_1(R) \to \pi_1(M)$ induced by the inclusion is injective, then by [Wa], $\psi_{n|R}$ is isotopic to the identity in S. If the map $i^* : \pi_1(R) \to \pi_1(M)$ is not injective, R contains a meridian. Since $\lambda \in \mathcal{D}(M)$, R must contain a component λ^i of λ and since γ has a maximal number of components, λ^i must be arational in R. Let us call r_n the restriction of s_n to R and suppose that the sequence (r_n) is not bounded in Teichmüller space. Since the length of ∂R is bounded, we can use Thurston's compactification and assume that (r_n) tends to a measured geodesic lamination ν. Since $l_{r_n}(\lambda_n \cap R) = l_\rho(\lambda_n \cap R) \leq l_{r_{n_0}}(\lambda_n \cap R) \to l_{r_{n_0}}(\lambda^i)$, $i(\nu, \lambda^i) = 0$ and ν and λ^i share the same support.

Let $m \subset R$ be a meridian. Then $m_n = \psi_n(m)$ is homotopic to m and therefore (m_n) is a sequence of meridians. We can assume that (m_n) converges in $\mathcal{PML}$ to a projective measured lamination represented by μ. Since $(\psi_n^* s_n)$ converges, then $l_{s_n}(m_n) = l_{\psi_n^* s_n}(\psi_n^{-1}(m_n)) = l_{\psi_n^* s_n}(m)$ converges and therefore $i(\mu, \nu) = 0$. Since ν and λ^i have the same support and since λ^i is arational in R, this implies that μ and λ^i have the same support. But the Casson's criterion (c.f. [Ot1], [Le1, Theorem B.1]) says that there exists a simple geodesic $l \subset R$ which is homoclinic and does not intersect μ transversely. This contradicts Lemma 3.6 and proves that the sequence (r_n) is bounded.

This applies to each component of $\partial M - \gamma$. It follows that we can choose the ψ_n such that each one is the composition of Dehn twists along the leaves of γ. We have seen above that the ψ_n are homotopic to the identity; by [Wa], each ψ_n can be extended to a homeomorphism of the whole manifold M. Let $\mathcal{V} \subset S$ be a small neighbourhood of γ; since $\lambda \subset \mathcal{D}(M)$, $\mathcal{V}$ does not contain the boundary of any essential annulus. It follows then from [Joh, Proposition 27.1] that, up to isotopy, each ψ_n has finite order. Since the ψ_n are compositions of Dehn twists along disjoint curves, they can not have finite order except when they are isotopic to the identity. We get from [CEG] that a subsequence of (f_n) converges to a pleated surface realizing λ. □

Let $f : S \to N$ be a pleated surface realizing a geodesic lamination L. Let $\mathbb{P}(N)$ be the tangent line bundle of N. We define a map $\mathbb{P}f$ from L to $\mathbb{P}(N)$ by mapping a point $x \in L$ to the direction of the unit vector tangent to $f(L)$ at $f(x)$.

The following injectivity theorem has been proved by Thurston ([Th2]) when M is boundary irreducible and by Otal ([Ot1]) when M is a compression body and $\lambda \in \hat{\mathcal{O}}$.

Theorem 5.5. *Let $\lambda \in \mathcal{D}(M)$ be a measured geodesic lamination not satisfying the condition $(-)$, let L be a geodesic lamination containing the support of λ and let $f : \partial M \to N$ be a pleated surface realizing L. Then the map from $\mathbb{P}f : L \to \mathbb{P}(N)$ is a homeomorphism into its image.*

Proof. Since the map f reduces the length, it is easy to see that $\mathbb{P}f$ is a continuous map and since L is compact, we need only to show that $\mathbb{P}f$ is injective.

Let us assume the contrary, there are two points u and $v \subset L$ such that $\mathbb{P}f(u) = \mathbb{P}f(v)$; let $\hat{f} : \mathbb{H}^2 \to \mathbb{H}^3$ be a lift of f and let $\hat{u}$ and $\hat{v}$ be lifts of u and v such that $\mathbb{P}\hat{f}(\hat{u}) = \mathbb{P}\hat{f}(\hat{v})$. Since $\hat{f}$ is an isometry on the preimage of L, it is injective on each leaf of the preimage of L. Therefore $\hat{u}$ and $\hat{v}$ lie in two different leaves $\hat{l}_1$ and $\hat{l}_2$ of the preimage of L. Since $\mathbb{P}\hat{f}(\hat{u}) = \mathbb{P}\hat{f}(\hat{v})$, then $\hat{f}(\hat{l}_1) = \hat{f}(\hat{l}_2)$. It follows that L is an annular lamination and since L does not intersect $\lambda \in \mathcal{D}(M)$ transversely, this contradicts Lemma 3.6. ☐

Remark. If λ satisfies the condition $(-)$, the same is true for λ but not for any geodesic lamination containing λ. ◇

6 Action on $\mathbb{R}$-trees

We will prove the following :

Proposition 6.1. *Let $\mathcal{T}$ be a real tree, let $\pi_1(M) \times \mathcal{T} \to \mathcal{T}$ be a small minimal action and let $\lambda \in \mathcal{D}(M)$ be a measured geodesic geodesic lamination. Then at least one connected component of λ is realized in $\mathcal{T}$.*

Proof. Let us first notice that this result has been proved by G. Kleineidam and J. Souto ([KlS1] and [KlS2]) when M is a compression body and λ lies in the Masur domain. The general case need just a reorganization of the proof of [Le1, Proposition 6]. Here we will sketch the proof which consists essentially in putting together ideas of [BoO] and of [KlS1].

If λ satisfies the condition $(-)$ then the elements of $\pi_1(M)$ corresponding to the leaves of λ form a generating subset of $\pi_1(M)$. In this case Proposition 6.1 is a straightforward consequence of [MoS1].

Let us assume that λ does not satisfies the condition $(-)$. For $c \in \pi_1(M)$ let us denote by $\delta_{\mathcal{T}}(c)$ the distance of translation of c on $\mathcal{T}$. Let S be a connected component of ∂M with $\chi(S) < 0$; the inclusion $i_* : \pi_1(S) \to \pi_1(M)$ provides us with an action of $\pi_1(S)$ on $\mathcal{T}$. By [MoO], there exists a measured geodesic lamination $\beta \in \mathcal{ML}(S)$ and a morphism $\phi : \mathcal{T}_\beta \to \mathcal{T}_S$ from the dual tree of β to the minimal subtree of $\mathcal{T}$ that is invariant by the action of $\pi_1(S)$. Since the action of $\pi_1(S)$ is not a priori small, ϕ is not, a priori, an isomorphism and there might be many laminations β with this property. We will consider such a lamination β which is adapted to our problem.

Let (λ_n) be a sequence of weighted multi-curves converging to λ in $\mathcal{ML}(\partial M)$ such that $(|\lambda_n|)$ converges to $|\lambda|$ in the Hausdorff topology. For each irrational sublamination λ^i of λ let us denote by $S(\lambda^i)$ the surface embraced by $|\lambda^i|$. For n large enough such that $|\lambda_n|$ does not intersect $\partial' \bar{S}(\lambda)$ transversely, let us add simple closed curves to $\partial' \bar{S}(\lambda) \cup |\lambda_n|$ in order to obtain a multi-curve L_n whose complementary regions are pairs of pants. By [MoO], there are measured geodesic laminations $\beta_n \in \mathcal{ML}(\partial M)$ and equivariant morphisms $\phi_n : \mathcal{T}_{\beta_n} \to \mathcal{T}$ such that for any leaf l_n of L_n, either $\delta_{\mathcal{T}}(l_n) > 0$ and the restriction of ϕ_n to the axis of l_n is an isometry or $\delta_{\mathcal{T}}(l_n) = 0$ and $i(l_n, \beta_n) = 0$, see [Le1, §4.1] for more details.

Extract a subsequence such that $(|\beta_n|)$ converges to a geodesic lamination B in the Hausdorff topology. The first step of the proof is to show that B intersects $|\lambda|$ transversely, this will allow us to follow [KlS1] by using a realization of a train track carrying λ to prove the proposition.

Lemma 6.2. *The geodesic lamination B intersects $|\lambda|$ transversely.*

Proof. The proof is done by contradiction; let us assume that $|\lambda|$ does not intersect B transversely.

If B is a multi-curve, then for large n, $|\beta_n| = B$ and β_n does not intersect λ transversely. By the definition of $\mathcal{D}(M)$, a small neighbourhood of B does not contain any essential disk, annulus or Moebius band. By [MoS1, Corollary IV 1.3], this implies that the action of $\pi_1(M)$ fixes a point of $\mathcal{T}$. This would contradict the assumption that this action is minimal.

Let us now consider the case where B is not a multi-curve. The first step in this case is to prove that $S(B)$ is incompressible for any connected component B^i of B. This will implies that a subsequence of $(|\beta_n|)$ is constant.

Claim 6.3. *If B does not intersects $|\lambda|$ transversely, then for any connected component B^i of B, the surface $S(B^i)$ is incompressible.*

Proof. Since we have assumed that B does not intersect $|\lambda|$ transversely, if B^i is a closed curve, the claim follows from the definition of $\mathcal{D}(M)$.

Let B^i be a component of B which is not a closed curve and let us assume that $S(B^i)$ contains a meridian. It follows from the ideas of [KlS1], that $S(B^i)$ contains a homoclinic leaf h which does not intersect B^i transversely (see [Le1, Lemma 4.3] for details). Since we have assumed that B does not intersect λ transversely, then $|\lambda| \cap S(B^i) \subset B^i$. Especially, h does not intersect λ transversely, contradicting Lemma 3.6. □

Let us explain how Claim 6.3 implies that for large n the support of β_n does not depend on n. Let B^i be a connected component of B; if B^i is a closed leaf then for large n, $B^i \subset |\beta_n|$. Let us next assume that B^i is not a closed leaf; by claim 6.3, $S(B^i)$ is incompressible, hence the action of $i_*(\pi_1(S(B^i))$ on its minimal subtree $\mathcal{T}_{S(B^i)} \subset \mathcal{T}$ is small. Since B does not intersect $\partial'\bar{S}(B^i)$, for large n, β_n does not intersect $\partial'\bar{S}(B^i)$. It follows that for each component d of $\partial'\bar{S}(B^i)$, the action of $i_*(d)$ has a fixed point in $\mathcal{T}_{S(B^i)}$. This allows us to apply Skora's theorem [Sk] which says that $\beta_n^i = \beta_n \cap S(B^i)$ is dual to the action of $i_*(\pi_1(S(B^i))$ on $\mathcal{T}_{S(B^i)}$. Doing this for each component of B, we obtain that, for large n, $|\beta_n|$ does not depend on n. Let us endow B with the measure of one of the β_n and let us call β the measured geodesic lamination thus obtained.

The last step in the proof of Lemma 6.2 is to show that $|\beta| = B$ is annular. Since we have assumed that B does not intersect $|\lambda|$ transversely, this will contradict the fact that $\lambda \in \mathcal{D}(M)$ (Lemma 3.6).

Claim 6.4. *The measured geodesic lamination β is annular*

Proof. By hypothesis β does not intersect λ transversely hence $S(\beta) \cap |\lambda| \subset |\beta|$.

Since $S(\beta)$ is incompressible, we might consider a characteristic submanifold W of $(M, S(\beta))$ (cf. [Joh] and [JaS]). Such a characteristic submanifold is a union of essential I-bundles and

Seifert fibered manifolds such that any essential annulus in $(M, S(\beta))$ can be homotoped in W. For each component Σ of $\partial M - S(B)$, $i_*(\Sigma)$ fixes a point in $\mathcal{T}$, hence by [Th4] (see also [MoS2, theorem IV 1.2]) W can be isotoped in such a way that we have $\beta \subset W \cap \partial M$.

We are considering the case where β is not a multi-curve, therefore it contains an irrational sublamination β^1. Since the Seifert fibered manifolds composing W intersect ∂M in annuli, $|\beta^1|$ lies in a component W^1 of W which is an essential I-bundle over a compact surface $F : W^1 = F \times I$. Let us denote by $p : F \times \partial I \to F$ the projection along the fibers. By Skora's theorem [Sk], for any component Σ of $W^1 \cap \partial M$, $\Sigma \cap \beta$ is dual to the action of $i_*(\pi_1(S))$ on $\mathcal{T}_\Sigma$. Since this action factorizes through the action of $\pi_1(W^1) = \pi_1(F)$, there is a measured geodesic lamination $\beta' \in \mathcal{ML}(F)$ such that $\beta \cap \partial W^1 \supset p^{-1}(\beta')$. Since the lamination $p^{-1}(\beta')$ is annular, β is annular (compare with [BoO, Lemma 14]). □

This claim concludes the proof of Lemma 6.2. □

Let us now complete the proof of Proposition 6.1. Let λ^i be a connected component of λ that intersects B transversely. Let us denote by $\pi_{\beta_n} : \mathbb{H}^2 \to \mathcal{T}_{\beta_n}$ the projection associated to the dual tree of β_n (as defined in §2.2). Since B intersects λ^i transversely, the construction in [Ot1, chap 3] yields a train track τ^i such that for large n, π_{β_n} is a weak realization of τ^i in $\mathcal{T}_{\beta_n}$.

Let l_n be a component of $L_n \cap S(\lambda^i)$. Up to extracting a subsequence, l_n converge in the Hausdorff topology to a geodesic lamination $L' \subset S(\lambda^i)$ that does not intersect λ^i transversely (by the choice of L_n). Therefore $|\lambda^i| \subset L'$. If up to extracting a subsequence, $i_*(l_n)$ has a fixed point in $\mathcal{T}$; then $i(\beta_n, l_n) = 0$. Letting n tends to ∞, we would get that B does not intersect $|\lambda^i|$ transversely, contradicting our choice of λ^i.

It follows from the previous paragraph that the restriction of ϕ_n to l_n is an isometry. For large n, each branch of $\hat{\tau}$ intersects transversely a lift of l_n. The fact that the restriction of ϕ_n to the axis of l_n is an isometry implies that $\phi_n \circ \pi_{\beta_n}$ is a weak realization of τ^i in $\mathcal{T}$ (compare with [KlS1, Lemma 11]). By [Ot1] this map $\phi_n \circ \pi_{\beta_n}$ is homotopic to a realization of λ^i in $\mathcal{T}$. □

Let $\rho_n : \pi_1(M) \to Isom(\mathbb{H}^3)$ be a sequence of representations containing no converging subsequence; in [MoS1], J. Morgan and P. Shalen described a way to associate a small minimal action of $\pi_1(M)$ on an $\mathbb{R}$-tree to some subsequence of (ρ_n). This can be stated in the following way : the sequence (ρ_n) tends to the action $\pi_1(M) \curvearrowright \mathcal{T}$ in the sense of Morgan and Shalen if there is a sequence $\varepsilon_n \longrightarrow 0$ such that for any $a \in \pi_1(M)$, $\varepsilon_n \delta_{\rho_n}(a) \longrightarrow \delta_\tau(a)$. In [Ot2], J.-P. Otal described, in the special case of handlebodies, the behavior of the length of measured geodesic laminations which are realized in $\mathcal{T}$. A careful look at the proof yields the following statement.

Theorem 6.5 (Continuity Theorem [Ot2]). *Let (ρ_n) be a sequence of discrete and faithful representations of $\pi_1(M)$ tending in the sense of Morgan and Shalen to a small minimal action of $\pi_1(M)$ on an $\mathbb{R}$-tree $\mathcal{T}$. Let $\varepsilon_n \longrightarrow 0$ be such that $\forall g \in \pi_1(M)$, $\varepsilon_n \delta_{\rho_n}(g) \longrightarrow \delta_{\mathcal{T}}(g)$ and let $L \subset \partial M$ be a geodesic lamination which is realized in $\mathcal{T}$. Then there exists a neighbourhood $\mathcal{V}(L)$ of L, and constants K, n_0 such that for any simple closed curve $c \subset \mathcal{V}(L)$ and for any $n \geq n_0$,*

$$\varepsilon_n l_{\rho_n}(c^*) \geq K l_{s_0}(c).$$

In the preceding statement s_0 is a fixed complete hyperbolic metric on $\partial_{\chi<0} M$. Using this and Proposition 6.1, we get the following

Theorem 6.6. *Let ρ_n be a sequence of faithful representations of $\pi_1(M)$ such that $\mathbb{H}^3/\rho_n(\pi_1(M))$ is homeomorphic to $int(M)$, let $\lambda \in \mathcal{D}(M)$ and let λ_n be a sequence of measured geodesic laminations such that :*

- *the sequence λ_n converges to λ in $\mathcal{ML}(\partial M)$;*

- *the sequence $|\lambda_n|$ converges to $|\lambda|$ in the Hausdorff topology;*

- *the sequence $l_{\rho_n}(\lambda_n)$ is bounded.*

Then (ρ_n) contains a converging subsequence.

Proof. Approximating each λ_n by weighted multi-curves, we produce a sequence of multi-curves also satisfying the hypothesis of the theorem. Let us assume that (ρ_n) does not contain an algebraically converging subsequence, then by [MoS1], a subsequence of (ρ_n) tends to a small minimal action of $\pi_1(M)$ on an $\mathbb{R}$-tree $\mathcal{T}$. By Proposition 6.1, λ is realized in $\mathcal{T}$ and it follows from Theorem 6.5 that $l_{\rho_n}(\gamma_n) \longrightarrow \infty$ giving us the desired contradiction. $\square$

Remark. When M is an I-bundle over a closed surface, the proof of this theorem can be found in [Th2]; this result has been extended to manifolds with incompressible boundary in [Oh1]. When M is a compression body and $\lambda \in \hat{\mathcal{O}}$, this result has been proved in [KlS1] and [KlS2]. $\diamond$

7 Conclusion

To complete this paper, we should also mention the action of $Mod(M)$ on $\mathcal{D}(M)$. The following result is proved in [Le2] using some properness properties of the bending map. The proof of these properties is long and is subject of [Le3]. Here we will only give an outline of the proof, the reader interested in a complete proof should refer to [Le2] or to [Le3].

Proposition 7.1. *If M is not a genus 2 handlebody, the action of $Mod(M)$ on $\mathcal{D}(M)$ is properly discontinuous.*

Outline of the proof. Here $Mod(M)$ is the group of isotopy classes of diffeomorphisms $M \to M$.

Let us assume that Proposition 7.1 is not true. There are measured geodesic laminations $\lambda \in \mathcal{D}(M)$, $(\lambda_n) \in \mathcal{D}(M)$ and diffeomorphisms $(\phi_n) \in Mod(M)$ such that (λ_n) and $(\phi_n(\lambda_n))$ converge to λ in $\mathcal{ML}(\partial M)$ and that for any $n \neq m$, ϕ_n is not isotopic to ϕ_m. Since $\lambda \in \mathcal{D}(M)$, $\exists \eta > 0$ such that $i(\lambda, \partial D) > \eta$ for any essential disc D. Let $\frac{2\pi}{\eta}\lambda$ be the measured geodesic lamination obtained by rescaling the measure of λ by $\frac{2\pi}{\eta}$. Let λ^i be a compact leaf of $\frac{2\pi}{\eta}\lambda$ with a weight greater than or equal to π; if, up to extracting a subsequence, λ^i is a compact leaf of all the measured geodesic laminations λ_n, let us replace, in $\frac{2\pi}{\eta}\lambda$ and in all $\frac{2\pi}{\eta}\lambda_n$, λ^i by a the same leaf with weight π. Let λ'_∞ and λ'_n be the measured geodesic laminations obtained by doing the same for all the leaves of $\frac{2\pi}{\eta}\lambda$ with a weight greater than π; let us remark that λ'_∞ may have some leaves with a weight greater than π but that for n large enough, the compact leaves of λ'_n have a weight less than or equal to π. Let us also remark that (λ'_n) and $(\phi_n(\lambda'_n))$ converge to λ'_∞ in $\mathcal{ML}(\partial M)$. By Lemma 3.5, λ'_∞ and λ'_n satisfy the conditions b), c). For n large enough, the λ'_n also satisfy the condition a) hence, by [Le1] (see also [Le2]), there is a geometrically finite metric ρ_n on the interior of M whose bending measured lamination is (λ'_n); here a geometrically finite metric is a geometrically finite representation $\rho : \pi_1(M) \to Isom(\mathbb{H}^3)$ together with an isotopy class of homeomorphisms $M \to N^{ep}$. The bending measured geodesic lamination of $\phi_{n*}(\rho_n)$ is $\phi_n(\lambda'_n)$ and by construction $\phi_n(\lambda'_n) \longrightarrow \lambda'_\infty$. It is at this point that we need the properness property of the bending map mentioned before the statement of Proposition 7.1 : it follows from [Le1] that there is a subsequence such that (ρ_n) and $(\phi_{n*}(\rho_n))$ converge to some geometrically finite metrics.

The conclusion comes from the fact that the action of $Mod(M)$ on the space of isotopy classes of geometrically finite metrics (see [Le3] for a definition) on the interior of M is properly discontinuous. This fact can be shown by using the arguments of the proof of the properness properties mentioned above (cf. [Le3]). $\square$

As has been mentioned throughout this paper, almost all the above results have been already proved when $\lambda \in \hat{\mathcal{O}}$. In an attempt to convince the reader of the interest of this paper we will give some examples of laminations lying in $\mathcal{D}$ but not in $\hat{\mathcal{O}}$.

Let M be an I-bundle over a compact surface S with boundary; this manifold M is a handlebody. Let $(\gamma, \alpha) \in \mathcal{ML}(S)$ be a pair of binding measured geodesic laminations, namely for any measured geodesic lamination $\beta \in \mathcal{ML}(S)$, $i(\beta, \gamma) + i(\beta, \alpha) > 0$. Such a pair of binding measured geodesic laminations has the following property : $\exists \eta > 0$ such that $i(c, \gamma) + i(c, \alpha) \geq \eta$ for any closed curve $c \subset S$. Let us defined a measured geodesic lamination $\lambda \in \mathcal{ML}(\partial M)$ as follows : on one component $\{0\} \times S$ of $\partial I \times S$, $\lambda \cap (\{0\} \times S)$ is γ, on the other component, $\lambda \cap (\{1\} \times S)$ is α and on the remaining part $I \times \partial S$ of the boundary, $\lambda \cap (I \times \partial S)$ is $\{p\} \times \partial S$ for some $p \in]0, 1[$ endowed with a Dirac mass η.

For any essential disc $D \subset M$, ∂D intersects $\{p\} \times \partial S$, hence $i(\partial D, \lambda) \geq \eta$. If A is an essential annulus, either ∂A intersects $\{p\} \times \partial S$ and $i(\partial A, \lambda) \geq \eta$, or A can be homotoped to a vertical annulus $c \times I \subset I \times S$ with c being a simple closed curve. In the second case, we have $i(\partial A, \lambda) = i(c, \gamma) + i(c, \alpha) \geq \eta$. We have thus proved that $\lambda \in \mathcal{D}(M)$. By [KlS1] the measured geodesic laminations $\lambda \cap \{0\} \times S$ and $\lambda \cap \{1\} \times S$ have the same supports as some measured laminations lying in $\mathcal{M}'$ hence $\lambda \notin \hat{\mathcal{O}}$.

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