LOCAL TOPOLOGY IN DEFORMATION SPACES OF HYPERBOLIC 3-MANIFOLDS II

JEFFREY F. BROCK, KENNETH W. BROMBERG, RICHARD D. CANARY, CYRIL LECUIRE, AND YAIR N. MINSKY

ABSTRACT. We prove that the deformation space $AH(M)$ of marked hyperbolic 3-manifolds homotopy equivalent to a fixed compact 3-manifold M with incompressible boundary is locally connected at quasiconformally rigid points.

1. Introduction

The space $AH(M)$ of (marked) hyperbolic 3-manifolds homotopy equivalent to a fixed compact orientable 3-manifold M plays an important role in the theory of 3-manifolds and is of interest in geometry and dynamics in general, by way of analogy with other parameter spaces such as those of conformal dynamical systems. The topology of this space at its boundary points is quite intricate and remains poorly understood after many years of study.

The main result of this paper is the following theorem proving local connectivity at a natural class of boundary points of $AH(M)$, whenever M has incompressible boundary. This generalizes previous work of Brock-Bromberg-Canary-Minsky [13], Ito [26] and Ohshika [44]. It can be contrasted with work of Bromberg [18] and Magid [34] showing that $AH(M)$ is not everywhere locally connected when M is an untwisted interval bundle.

Theorem 1.1. *If M is a compact, hyperbolizable orientable 3-manifold with incompressible boundary and $\tau \in AH(M)$ is quasiconformally rigid, then $AH(M)$ is locally connected at τ.*

A hyperbolic 3-manifold N is *quasiconformally rigid* if any hyperbolic 3-manifold which is bilipschitz homeomorphic to N is actually isometric to N. Equivalently, N is quasiconformally rigid if every component of its conformal boundary is a thrice-punctured sphere. Note that the conformal boundary may also be empty. See Section 2 for detailed definitions.

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Bromberg [18] showed that the space of Kleinian punctured torus groups fails to be locally connected at some, but not all, points whose conformal boundary consists of one once-punctured torus and two thrice-punctured spheres. Therefore, one cannot expect any better result whose assumptions simply restrict the complexity of the conformal boundary.

Our argument actually establishes the stronger fact that components of the interior of $AH(M)$ cannot *self-bump* at a quasiconformally rigid point. (We say that a component C of the interior of $AH(M)$ self-bumps at ρ if any sufficiently small neighborhood of ρ in $AH(M)$ has disconnected intersection with C.)

Theorem 1.2. *Let M be a compact, hyperbolizable 3-manifold with incompressible boundary. If $\tau \in AH(M)$ is quasiconformally rigid, then no component of* $\text{int}(AH(M))$ *self-bumps at τ.*

History and prior results

We recall briefly the history of the subject. The interior of $AH(M)$ is well-understood due to the work of Bers [6] and others. The components of the interior are enumerated by marked homeomorphism types and each component is parameterized by analytic data (see [23, Chapter 7] for a survey of this theory in topological language). The entire space $AH(M)$ is the closure of its interior (see [41, 43, 16, 11, 17]). The Ending Lamination Theorem [39, 16, 9] provides a classification of hyperbolic 3-manifolds in $AH(M)$ in terms of ending invariants which record the asymptotic geometry, but these ending invariants vary discontinuously (see [1, 10]).

We will focus on the case where M has incompressible boundary. Anderson and Canary [1] discovered that components of the interior of $AH(M)$ can *bump*, i.e. have intersecting closure, and Anderson, Canary and McCullough [3] characterized which components can bump in terms of the topological change of homeomorphism type involved. McMullen [38], in the untwisted interval bundle case, and Bromberg and Holt [19], more generally, discovered that components of the interior of $AH(M)$ can self-bump. Bromberg [18] and Magid [34] showed that $AH(M)$ fails to be locally connected when M is an untwisted interval bundle and Bromberg conjectures that $AH(M)$ is never locally connected. See [21] for a more complete discussion of the topology of $AH(M)$.

In an earlier paper [13] we showed that there is no bumping or self-bumping at a representation $\rho \in AH(M)$ if $\rho(\pi_1(M))$ contains no parabolic elements which do not lie in rank two abelian subgroups of $\rho(\pi_1(M))$. If there is no bumping or self-bumping at ρ then we say that ρ is *uniquely approachable* and we notice that $AH(M)$ is locally connected at uniquely approachable representations. We further showed that there is no bumping at quasiconformally rigid points ([13, Thm. 1.2]), and that if M is either an untwisted interval bundle or an acylindrical 3-manifold, then quasiconformally rigid points are uniquely approachable ([13, Cor. 1.4]). Theorem 1.1 extends these results to the general incompressible-boundary case. We note that Ito [26] and Ohshika [44] obtained related results in the setting of quasifuchsian groups.

Sketch of proof

The geometry of a hyperbolic 3-manifold is determined by its *end invariants*. However, since these invariants do not vary continuously over $AH(M)$, one must be extremely careful when using them to understand the topology of $AH(M)$.

Let τ be a quasiconformally rigid point on the boundary of a component C of $\mathrm{int}(AH(M))$. We may assume that the quotient manifold N_τ is homeomorphic to $\mathrm{int}(M)$, so the end invariants of τ consist of a curve system, called the *parabolic locus*, p_τ on ∂M, associated to the cusps of N_τ, and a lamination λ_W filling each component W of $\partial M \setminus p_\tau$ which is not a 3-holed sphere. Since we have previously shown that there is no bumping at τ, our goal is to show that C does not self-bump at τ. Assume for simplicity that M is not an interval bundle, since there will be some additional complications in this case.

Neighborhood system. Given $\rho \in C$, the ending data of ρ is a conformal structure on ∂M and we can consider its projection $\pi_W(\rho)$ to the *curve complex* $\mathcal{C}(W)$ of each component W of $\partial M \setminus p_\tau$. We define a "neighborhood" for τ in C by choosing a neighborhood in each $\mathcal{C}(W)$ of the ending lamination component λ_W and requiring the projections $\pi_W(\rho)$ lie there, and also placing a bound on the length $\ell_\rho(\alpha)$ in N_ρ of each component α of p_τ. Proposition 3.1 shows that a sequence of representations in C converges to τ if and only it eventually lies in every such "neighborhood." Equivalently, we say that sets of this form are the intersection with C of a neighborhood system for τ in $AH(M)$. When M is acylindrical, we established this in [13, Lemma 8.2].

The first new ingredient in our more general situation is a compactness theorem of Lecuire [33] (Theorem 3.3), which is applied to show that a sequence of representations eventually contained in any neighborhood of the above form must have a convergent subsequence. If ρ is a limit of such a sequence, one applies a result on convergence of ending invariants ([14, Thm. 1.3]) to exhibit a homotopy equivalence from N_τ to N_ρ which takes cusps to cusps and geometrically infinite ends to geometrically infinite ends with the same ending laminations. A topological observation of Canary and Hersonsky [22, Prop. 8.1] allows one to upgrade this homotopy equivalence to a homemorphism and we conclude that $\tau = \rho$ by applying the Ending Lamination Theorem [16]. The converse, that any sequence in C converging to τ eventually lies in any neighborhood of the form above, follows nearly immediately from [14, Thm. 1.3].

Skinning map and control of subsurface projections. If $M = S\times[0,1]$ for a closed surface S, we use the shorthand $AH(S)$ for $AH(S\times[0,1])$. The end invariant of a Kleinian surface group $\rho \in AH(S)$ consists of the end invariant $\nu_+(\rho)$ on the top component $S \times \{1\}$ of $\partial(S \times [0,1])$ and the end invariant $\nu_-(\rho)$ of the bottom component $S \times \{0\}$. Given a curve α on S, we define a quantity $\mathbf{m}_\alpha(\nu_+(\rho), \nu_-(\rho))$ (see Section 2) which depends on the length of α in $\nu_+(\rho)$ and $\nu_-(\rho)$ and the distances $d_W(\nu_+(\rho), \nu_-(\rho))$ between projections of $\nu_+(\rho)$ and $\nu_-(\rho)$ onto subsurfaces bordering α. It follows from work of Minsky [39] that $\mathbf{m}_\alpha(\nu_+(\rho), \nu_-(\rho))$ is "large" if and only if $\ell_\rho(\alpha)$ is "small" (Theorem 2.3). This allows us to determine membership in the above neighborhood system by studying the quantity $\mathbf{m}_\alpha$.

When $\rho^S = \rho|_{\pi_1(S)}$ is the restriction of ρ to the fundamental group of a boundary component S of M, and with suitable orientation convention, $\nu_+(\rho^S)$ is just the restriction of the ending invariant of ρ to S. The other ending invariant $\nu_-(\rho^S)$ is determined more subtly — it is the image by Thurston's *skinning map* of the full ending invariant of ρ. Thus the control needed for determining if a point is in the aforementioned neighborhoods of τ requires some control over the skinning map. In the acylindrical case, Thurston's Bounded Image Theorem provided this control.

In Proposition 4.1 we show that if α is one of the curves in $p_\tau \cap S$ and ρ is close enough to τ, then the contribution of $\nu_-(\rho^S)$ to $\mathbf{m}_\alpha(\nu_+, \nu_-)$ is bounded from above. More explicitly, if we fix a marking μ on S and define $\mathbf{m}_\alpha(\mu, \nu_-)$ similarly to $\mathbf{m}_\alpha(\nu_+, \nu_-)$, then $\mathbf{m}_\alpha(\mu, \nu_-(\rho^S))$ is bounded above in some neighborhood of τ. Proposition 4.1 is a consequence of Theorem 2.4, a result on convergence of Kleinian surface groups proven by Brock-Bromberg-Canary-Lecuire [12].

Navigation in Teichmüller space. The last main hurdle in establishing local connectivity is to show that, within such neighborhoods, one can find paths in which the projection data can be sufficiently well controlled. The tool for this is provided by Lemma 5.2, which was proved in [13, Lemma 6.1]. With this in hand we rule out self-bumping by showing that for each neighborhood $\mathcal{U}$ of τ there is a smaller neighborhood $\mathcal{U}'$ such that any two points in $\mathcal{U}'$ can be connected by a path in $\mathcal{U}$, see Proposition 5.1.

2. Background

2.1. Hyperbolic 3-manifolds and their deformation spaces

We will assume throughout the paper that M is a compact, orientable, hyperbolizable 3-manifold with non-empty incompressible boundary which is not an interval bundle over the torus. We recall that M is *hyperbolizable* if its interior admits a complete metric of constant negative curvature -1 and that M has *incompressible boundary* if whenever S is a component of the boundary ∂M of M, then S is not a sphere and the inclusion map of S into M induces an injection of $\pi_1(S)$ into $\pi_1(M)$. Let $\partial_0 M$ denote the collection of non-toroidal components of ∂M.

Let $AH(M)$ denote the space of (conjugacy classes of) discrete, faithful representations of $\pi_1(M)$ into $\mathrm{PSL}(2, \mathbb{C})$. We view $AH(M)$ as a quotient of a subset of the space $\mathrm{Hom}(\pi_1(M), \mathrm{PSL}(2,\mathbb{C}))$ of homomorphisms of $\pi_1(M)$ into $\mathrm{PSL}(2,\mathbb{C})$. We give $\mathrm{Hom}(\pi_1(M), \mathrm{PSL}(2,\mathbb{C}))$ the compact-open topology and $AH(M)$ inherits the induced quotient topology. In the case that $M = S \times [0,1]$ and S is a closed oriented surface, we use the shorthand $AH(S)$ for $AH(S \times [0,1])$.

If $\rho \in AH(M)$, then $N_\rho = \mathbb{H}^3/\rho(\pi_1(M))$ is a complete hyperbolic 3-manifold. There exists a universal constant $\mu > 0$, called the *Margulis constant*, so that if $\epsilon < \mu$, then every component of

$$(N_\rho)_{(0,\epsilon)} = \{x \in N_\rho \mid \mathrm{inj}(x) < \epsilon\}$$

(where $\mathrm{inj}(x)$ is the injectivity radius of N_ρ at x), is either a *Margulis tube*, i.e. an open solid torus neighborhood of a closed geodesic in N_ρ or a *cusp*, i.e. a quotient of a horoball H in $\mathbb{H}^3$ by a group of parabolic elements of $\rho(\pi_1(M))$ which preserve H. We fix $\epsilon_0 < \mu$ and let N_ρ^0 be obtained from N_ρ by removing the cusps in $(N_\rho)_{(0,\epsilon_0)}$. A *relative compact core* M_ρ for N_ρ^0 is a compact submanifold of N_ρ^0 so that the inclusion of M_ρ into N_ρ is a homotopy equivalence and if $P_\rho = M_\rho \cap \partial N_\rho^0$ and R is a component of ∂N_ρ^0, then the inclusion of $R \cap P_\rho$ into R is a homotopy equivalence (see Kulkarni-Shalen [32] or McCullough [36] for the existence of relative compact cores). Bonahon [7] showed that N_ρ is homeomorphic to the interior $\mathrm{int}(M_\rho)$ of M_ρ and that each component of $N_\rho^0 - \mathrm{int}(M_\rho)$ is bounded by a component F of $\partial M_\rho - \mathrm{int}(P_\rho)$ and is homemorphic to $F \times [0,\infty)$.

There is a homotopy equivalence $h_\rho : M \to M_\rho$, well-defined up to homotopy, in the homotopy class determined by ρ. It will often be natural to restrict to the subset $AH_0(M)$ where h_ρ is homotopic to an orientation-preserving homeomorphism. If $h : M \to M'$ is a homotopy equivalence, then h induces an identification of $AH(M)$ with $AH(M')$. In particular, if $\rho \in AH(M)$, then we may identify $AH_0(M)$ with $AH(M_\rho)$, so that $\rho \in AH_0(M_\rho)$. So, in our study of the local topology of $AH(M)$, it does not reduce generality to assume that we are in a neighborhood of a representation in $AH_0(M)$. Moreover, a quasiconformally rigid representation in $AH_0(M)$ has a neighborhood in $AH(M)$ which is entirely contained in $AH_0(M)$, see Theorem 2.1 below.

2.2. **Ending invariants**

If W is a compact orientable hyperbolizable surface with boundary, other than the disk or annulus, then the vertex set of its *curve complex* $\mathcal{C}(W)$ is the set of (isotopy classes of) simple, closed, non-peripheral curves on W. If W is not a one-holed torus or a four-holed sphere, a collection $\{\alpha_0, \alpha_1, \dots, \alpha_n\}$ of vertices of $\mathcal{C}(W)$ span a n-simplex in $\mathcal{C}(W)$ if and only if the (isotopy classes of) curves have mutually disjoint representatives. Masur and Minsky [35] proved that $\mathcal{C}(W)$ is Gromov hyperbolic and Klarreich [30] (see also Hamenstädt [24]) identified its Gromov boundary $\partial_\infty \mathcal{C}(W)$ with the space of filling geodesic laminations on W which are the support of a measured lamination. Recall that, given a background finite area metric on W, a *geodesic lamination* is a closed subset which is a disjoint union of geodesics. A geodesic lamination is *filling* if it intersects every closed geodesic on W essentially. A *measured lamination* is a geodesic lamination together with a transverse measure, i.e. an assignment of a measure to each arc transverse to the lamination which is invariant under isotopies preserving the lamination.

Let p_ρ be the core curves of the annular components of $P_\rho = M_\rho \cap N^0_\rho$. Then p_ρ is called the *parabolic locus* of ρ and is a well-defined isotopy class of curves in $\partial_0 M_\rho$.

Let $\Omega(\rho)$ denote the largest open subset of $\partial_\infty \mathbb{H}^3 = \widehat{\mathbb{C}}$ which $\rho(\pi_1(M))$ acts properly discontinuously on. Then

$$\partial_c N_\rho = \Omega(\rho)/\rho(\pi_1(S))$$

is a Riemann surface, called the *conformal boundary*, and $\widehat{N}_\rho = N_\rho \cup \partial_c N_\rho$ is a 3-manifold with boundary. We say that ρ is *quasiconformally rigid* if every component of the conformal boundary is a thrice-punctured sphere.

If W is a component of $\partial_0 M_\rho - p_\rho$, then either W is parallel to a component of $\partial_c N_\rho$, in $\widehat{N_\rho} - M_\rho$, in which case we say that W bounds a geometrically finite end, and one obtains a well-defined finite area conformal structure on W from the parallel component of $\partial_c N_\rho$, or W bounds a geometrically infinite end. If W bounds a geometrically infinite end, then there exists a sequence $\{\alpha_n\}$ of simple closed curves on W whose geodesic representatives $\{\alpha_n^*\}$ in N_ρ intersect the component of $N_\rho - M_\rho$ bounded by $h_\rho(W)$ and exit every compact subset of N_ρ. In this case, the sequence $\{\alpha_n\}$ converges to a point in $\partial_\infty \mathcal{C}(W)$, which is called the *ending lamination* of W in N^0_ρ.

The *ending invariant* $\nu(\rho)$ of ρ consist of the parabolic locus p_ρ on M_ρ, the conformal structures on components of $\partial_0 M_\rho - p_\rho$ which bound a geometrically finite end, and the ending laminations on the components which bound geometrically infinite ends. The Ending Lamination Theorem [39, 16, 9] implies that if $\rho, \sigma \in AH(M)$ and there exists a homeomorphism $g : M_\rho \to M_\sigma$, in the homotopy class of $\sigma \circ \rho^{-1}$ which takes the end invariants of ρ to the end invariants of σ, then there exists an isometry $G : N_\rho \to N_\sigma$ in the homotopy class of $\sigma \circ \rho^{-1}$ which takes ends in N^0_ρ to the ends of N^0_σ with the corresponding ending data. See the discussion in Minsky [39, Sec. 2] for more details on ending invariants.

2.3. Topology of deformation spaces

The topology of the interior $\text{int}(AH(M))$ of $AH(M)$ is well understood. Sullivan [48] proved that $\rho \in AH(M)$ lies in the interior of $AH(M)$ if and only if the ending invariants consist entirely of conformal structures on non-toroidal boundary components of ∂M_ρ, i.e. N^0_ρ has no geometrically infinite ends or annular boundary components corresponding to rank one cusps. If $\rho \in \text{int}(AH_0(M))$, then we may identify M_ρ with M and view the end invariant $\nu(\rho)$ as an element of $\mathcal{T}(\partial_0 M)$. Work of Bers [6] implies that this identification induces a homeomorphism between $\text{int}(AH_0(M))$ and $\mathcal{T}(\partial_0 M)$. In general, if ρ lies in a component C of $\text{int}(AH(M))$, one obtains an identification of C with $\mathcal{T}(\partial M_\rho)$, but M_ρ need not be homeomorphic to M (see [23] for a more detailed discussion of this parameterization and its history). Brock, Canary and Minsky [16] proved that when M has incompressible boundary, then $AH(M)$ is the closure of $\text{int}(AH(M))$, and Namazi-Souto [41] and Ohshika [43] established the same fact when M is any compact hyperbolizable manifold. However, Anderson and Canary [1] showed that the closure of $\text{int}(AH_0(M))$ need not be entirely contained in $AH_0(M)$.

In our earlier paper, we investigated the "bumping locus" of $AH(M)$ and showed that $AH(M)$ cannot bump at a quasiconformally rigid point ρ, i.e. a quasiconformally rigid point is in the closure of exactly one component of $\text{int}(AH(M))$. (In fact, the result there holds without the restriction that M have incompressible boundary.)

Theorem 2.1. ([13, Thm. 1.2 and Prop. 3.2]) *If M is a compact, orientable hyperbolizable 3-manifold with incompressible boundary and $\rho_0 \in \partial AH(M)$ is quasiconformally rigid, then $AH(M)$ cannot bump at ρ. Moreover, there exists a neighborhood U of ρ_0 in $AH(M)$, so that if $\rho \in U$, then there exists an orientation-preserving homeomorphism $h : N_\rho \to N_{\rho_0}$ in the homotopy class of $\rho_0 \circ \rho^{-1}$.*

We will make crucial use of a key tool in the proof of this result. Recall that a sequence $\{\Gamma_n\}$ of Kleinian groups is said to converge *geometrically* to a Kleinian group Γ if $\{\Gamma_n\}$ converges to Γ in the Chabauty topology on closed subsets of $\text{PSL}(2, \mathbb{C})$. If a sequence $\{\rho_n\}$ converges to ρ in $AH(M)$, then we may choose representatives in $\text{Hom}(\pi_1(M), \text{PSL}(2, \mathbb{C}))$, still called $\{\rho_n\}$ and ρ, so that $\{\rho_n\}$ converges to ρ in $\text{Hom}(\pi_1(M), \text{PSL}(2, \mathbb{C}))$ and $\{\rho_n(\pi_1(M))\}$ converges geometrically to a torsion-free Kleinian group $\hat{\Gamma}$ which contains $\rho(\pi_1(M))$, see Jørgenson-Marden [29, Prop. 3.8]. In this situation, there is a natural covering map from N_ρ to $\hat{N} = \mathbb{H}^3/\hat{\Gamma}$. If ρ is quasiconformally rigid, then this covering map restricts to an embedding on some relative compact core for N^0_ρ:

Proposition 2.2. (Anderson-Canary-Culler-Shalen [2, Prop. 3.2, Remark 3.3]) *If ρ is a quasiconformally rigid point in $\partial AH(M)$ and $\{\rho_n\}$ converges to ρ and $\{\rho_n(\pi_1(M))\}$ converges geometrically to $\hat{\Gamma}$, then there is a compact core M_ρ for N_ρ which embeds in $\widehat{N} = \mathbb{H}^3/\hat{\Gamma}$ under the obvious covering map.*

Remark: The results of [2] actually show that the convex core $C(N_\rho)$ of N_ρ embeds in $\widehat{N}$. One may then simply take M_ρ to be a compact core for $C(N_\rho)$, which is hence a compact core for N_ρ, as in the proof of [13, Prop. 3.2], to obtain our Proposition 2.2.

2.4. **Subsurface projections and Kleinian surface groups**

It is a central ingredient in the proof of the Ending Lamination Theorem that the subsurface projections of the ending invariants of a Kleinian surface group $\rho \in AH(S)$ coarsely determine the geometry of N_ρ. We recall several explicit forms of this crucial principle.

If $\rho \in AH(S)$, then its end invariant decomposes as a pair $(\nu_+(\rho), \nu_-(\rho))$ where $\nu_+(\rho)$ is the ending invariant of the upward pointing end of N_ρ. We recall that $AH(S) = AH_0(S \times [0,1])$ and that there exists an orientation-preserving homeomorphism $h_\rho : S \times [0,1] \to M_\rho$ and we call $h_\rho(S \times [0,1])$ the upward-pointing component of ∂M_ρ and call its associated end invariant upward-pointing. Curves in the parabolic locus of the upward-pointing end invariant are called upward-pointing parabolic curves.

If W is an essential subsurface of S, one may (coarsely) define a subsurface projection $\pi_W : \mathcal{E}(S) \to \mathcal{C}(W)$ where $\mathcal{E}(S)$ is the collection of possible ending invariants on S. If W is not an annulus and $\nu \in \mathcal{T}(S)$ is a conformal structure on S, then $\pi_W(\nu)$ is obtained by considering a shortest curve, in the induced hyperbolic structure on S, which essentially intersects S and surgering it with ∂W to obtain an element of $\mathcal{C}(W)$. There are several choices involved in this construction, but there is a uniform upper bound on the distance between any two such curves, and we simply choose one of the curves obtained in this manner. One must take more care in the case that ν is a general ending invariant or W is an incompressible annulus in S, see [39, Section 4] for a complete discussion.

If α is a simple closed curve on S, and ν and μ are ending invariants in $\mathcal{E}(S)$, we define

$$\mathbf{m}_\alpha(\nu,\mu) = \max\left\{\frac{1}{\ell_\alpha(\nu)}, \frac{1}{\ell_\alpha(\mu)}, \sup_{\alpha\subset\partial W} d_W(\nu,\mu)\right\}$$

where

$$d_W(\nu,\mu) = d_{\mathcal{C}(W)}(\pi_W(\nu) \cup \pi_W(\mu))$$

and the supremum in the final term is taken over all essential subsurfaces W with α in their boundary. Moreover, if ν is an ending invariant on S, we define $\ell_\alpha(\nu)$ to be the length of the geodesic representation of α in the associated hyperbolic structure if α lies in a component of the complement of the parabolic locus p_ν of ν which admits a conformal structure, we define $\frac{1}{\ell_\alpha(\nu)} = +\infty$ if α lies in the parabolic locus p_ν of ν and we define $\frac{1}{\ell_\alpha(\nu)} = 0$ otherwise.

The Length Bound Theorem from [16] shows that the end invariants coarsely determine the set of "short" curves in N_ρ and their length. The following simplified version of the Length Bound Theorem is a mild generalization of Theorem 2.2 in our previous paper [13].

Theorem 2.3. ([16]) *Suppose that S is a compact, oriented, hyperbolic surface and $\rho \in AH(S)$.*

(1) Given $\delta > 0$, there exists $K = K(\delta, S)$, so that if $\mathbf{m}_\alpha(\nu_+(\rho), \nu_-(\rho)) > K$, then $\ell_\alpha(\rho) < \delta$.
(2) Given $K > 0$, there exists $\epsilon = \epsilon(K,S) > 0$ so that if $\ell_\alpha(\rho) < \epsilon$, then $\mathbf{m}_\alpha(\nu_+(\rho), \nu_-(\rho)) > K$.

The results of [12] give a relatively complete picture of the relationship between the asymptotic behavior of the ending invariants of a convergent sequence of Kleinian surface groups and the geometry of the algebraic limit. We state results from [12] in the simpler case where one knows that a compact core of the algebraic limit embeds in the geometric limit. Proposition 2.2 will assure us that we are always in this simpler case when considering a sequence which converges to a quasiconformally rigid point.

Recall that a *complete marking* of S is a maximal collection $\{\alpha_1, \dots, \alpha_i\}$ of disjoint simple closed curves on S, together with a collection $\{\beta_1, \dots, \beta_i\}$ of simple closed curves such that β_i is disjoint from α_j if $j \neq i$ and intersects α_i minimally. If μ is a complete marking of S and W is an essential subsurface of S, then we may obtain $\pi_W(\mu) \in \mathcal{C}(W)$ by surgering a curve in μ which intersects W essentially with ∂W. Again, the annulus case is slightly more complicated, see [39, Section 5.1] for more details. If $\nu \in \mathcal{E}(S)$ and μ is a complete marking on S, then we define

$$\mathbf{m}_\alpha(\nu, \mu) = \max\left\{\frac{1}{\ell_\alpha(\nu)}, \sup_{\alpha \subset \partial W} d_W(\nu, \mu)\right\}$$

where $d_W(\nu, \mu) = d_{\mathcal{C}(W)}(\pi_W(\nu), \pi_W(\mu))$ and the supremum in the final term is taken over all essential subsurfaces W with α in their boundary.

It is an elementary exercise to verify that $\mathbf{m}_\alpha$ satisfies the triangle inequality.

The following result combines portions of the two main results of [12] in our simpler setting. (Notice that Lemmas 4.3 and 4.4 in [12] assure that we do not have to pass to a further subsequence as in the statements of Theorems 1.1 and 1.2 in [12].)

Theorem 2.4. ([12, Thm. 1.1, Thm. 1.2]) *Suppose that $\{\rho_n\}$ is a sequence in $AH(S)$ converging to $\rho \in AH(S)$, $\{\rho_n(\pi_1(S))\}$ converges geometrically to $\hat\Gamma$ and there is a compact core for N_ρ that embeds in $\widehat{N} = \mathbf{H}^3/\hat\Gamma$. If α is an upward-pointing parabolic for ρ and μ is a complete marking on S, then $\{\mathbf{m}_\alpha(\nu_-(\rho_n), \mu)\}$ is eventually bounded. In any case, $\ell_\rho(\alpha) > 0$ if and only if $\{\mathbf{m}_\alpha(\nu_+(\rho_n), \mu)\}$ and $\{\mathbf{m}_\alpha(\nu_-(\rho_n), \mu)\}$ are both eventually bounded.*

The following result from [14] allows us to control the development of geometrically infinite ends and the resulting ending laminations.

Theorem 2.5. ([14, Thm. 1.3]) *Suppose that $\{\rho_n\}$ is a sequence in $AH(S)$ converging to $\rho \in AH(S)$. If $W \subseteq S$ is an essential subsurface of S, other than an annulus or a pair of pants, and $\lambda \in \mathcal{EL}(W)$ is a lamination supported on W, the following statements are equivalent:*

(1) λ is a component of $\nu_+(\rho)$.
(2) $\{\pi_W(\nu_+(\rho_n))\}$ converges to λ.

3. Neighborhood systems for quasiconformally rigid points

In this section, we produce a neighborhood system for a quasiconformally rigid representation $\tau \in AH_0(M)$. We recall that Theorem 2.1 implies that τ lies in the boundary of $\mathrm{int}(AH_0(M))$ and lies in the boundary of no other component of $\mathrm{int}(AH(M))$.

Let $\mathcal{W}_\tau$ denote the collection of components of $\partial_0 M_\tau - P_\tau$ which are not thrice-punctured spheres. (Here, and throughout, we identify M_τ with M and hence P_τ with a collection of incompressible annuli and tori in $\partial_0 M$.) Let p_τ denote the multicurve of cores of annulus components of P_τ. If $W \in \mathcal{W}_\tau$, since τ is quasiconformally rigid the end invariant associated to W is an ending lamination $\lambda_W \in \partial_\infty \mathcal{C}(W)$. Given $\delta > 0$ and a collection $\mathbb{U} = \{U_W\}_{W \in \mathcal{W}_\tau}$ so that each U_W is a neighborhood of $\lambda_W \in \partial_\infty \mathcal{C}(W)$ in $\mathcal{C}(W)$, we define $\mathcal{U}(\delta, \mathbb{U}, \tau)$ to be the set of all $\rho \in \mathrm{int}(AH_0(M))$ such that

(1) $\ell_\rho(\alpha) < \delta$ if $\alpha \in p_\tau$, and
(2) $\pi_W(\nu(\rho)) \in U_W \in \mathbb{U}$ if $W \in \mathcal{W}_\tau$.

If τ is a maximal cusp, meaning that p_τ is a pants decomposition of $\partial_0 M$, then $\mathcal{W}_\tau = \emptyset$ and hence $\mathbb{U} = \emptyset$, and we also write $\mathcal{U}(\delta, \tau) = \mathcal{U}(\delta, \emptyset, \tau)$.

If $\rho \in AH(M)$, then $\bar{\rho} \in AH(M)$ is obtained from ρ by complex conjugation, i.e. conjugation by $z \to \bar{z}$. Notice that $N_{\bar{\rho}}$ is simply N_ρ with the opposite orientation. One may check that $\rho = \bar{\rho}$ if and only if M is an interval bundle and ρ is virtually Fuchsian i.e. $\rho(\pi_1(M))$ has a finite index subgroup conjugate into $\mathrm{PSL}(2, \mathbb{R})$ (but we will not use this fact).

If M is an interval bundle, then there exists an orientation-reversing involution $\iota_M : M \to M$ which preserves each fiber and is homotopic, but not isotopic, to the identity. In this case, $AH(M) = AH_0(M)$, and if ν is the end invariant for $\rho \in AH(M)$, then $i_M(\nu)$ is the end invariant for $\bar{\rho}$.

The following proposition is the main result of this section, stating that sets of the form $\mathcal{U}(\delta, \mathbb{U}, \tau)$ form a neighborhood system for τ (in the case that M is an interval bundle and τ is a maximal cusp, we actually obtain a neighborhood system for the pair $\{\tau, \bar{\tau}\}$).

Proposition 3.1. *Suppose that M has incompressible boundary and τ is a quasiconformally rigid representation in $AH_0(M)$. If M is not an interval bundle or $\mathcal{W}_\tau$ is non-empty, then the collection of sets of the form $\mathcal{U}(\delta, \mathbb{U}, \tau)$ is the intersection of a local neighborhood system for τ in $AH(M)$ with* $\mathrm{int}(AH_0(M))$.

If M is an interval bundle and $\mathcal{W}_\tau$ is empty, then the collection of sets of the form $\mathcal{U}(\delta, \tau)$ is the intersection with $\mathrm{int}(AH_0(M))$ *of a local neighborhood system for $\{\tau, \bar{\tau}\}$ in $AH(M)$.*

It suffices to prove that a sequence $\{\rho_n\}$ in $\mathrm{int}(AH_0(M))$ converges to τ (or accumulates on $\{\tau, \bar{\tau}\}$ when M is an interval bundle and $\mathcal{W}_\tau$ is empty) if and only if it is eventually contained in any set of the form $\mathcal{U}(\delta, \mathbb{U}, \tau)$.

One direction follows easily from Theorem 2.5. Notice that if M is an interval bundle and $\mathcal{W}_\tau$ is empty, then $\mathcal{U}(\delta, \tau) = \mathcal{U}(\delta, \bar{\tau})$ for all $\delta > 0$.

Lemma 3.2. *Suppose that M has incompressible boundary, τ is a quasiconformally rigid representation in $AH_0(M)$ and $\{\rho_n\} \subset \text{int}(AH_0(M))$ converges to τ. If $\delta > 0$ and $\mathbb{U} = \{U_W\}_{W\in\mathcal{W}_\tau}$ is a collection of neighborhoods of the ending laminations of N^0_τ, then ρ_n is contained in $\mathcal{U}(\delta, \mathbb{U}, \tau)$ for all sufficiently large n.*

Proof. If $\alpha \in p_\tau$, then $\ell_\alpha(\tau) = 0$, so, since $\lim \ell_\alpha(\rho_n) = \ell_\alpha(\tau)$, $\ell_\alpha(\rho_n) < \delta$ for all sufficiently large n. Fix $W \in \mathcal{W}_\tau$ and let S be the component of ∂M containing W. Let $\rho^S_n = \rho_n|_{\pi_1(S)}$ and $\tau^S = \tau|_{\pi_1(S)}$. Since $\{\rho^S_n\}$ converges to τ^S in $AH(S)$, and the geometrically infinite end associated to W is upward-pointing, Theorem 2.5 implies that $\{\pi_W(\nu(\rho_n))\}$ converges to λ_W. (Recall that with this convention $\nu_+(\rho^S_n)$ is the restriction of $\nu(\rho_n)$ to S.) In particular, $\pi_W(\nu(\rho_n))$ is contained in $U_W \in \mathbb{U}$ for all sufficiently large n. Therefore, ρ_n is contained in $\mathcal{U}(\delta, \mathbb{U}, \tau)$ for all sufficiently large n. □

In order to establish the other direction of our claim, we will need the following (slight generalization of a) criterion for convergence due to Bonahon and Otal [8, Lem. 14]. This criterion is a common generalization of Thurston's Double Limit Theorem [50] and Relative Boundedness Theorem [51, Thm 3.1]. It was generalized to manifolds with compressible boundary by Kleineidam-Souto [31] and Lecuire [33, Thm. 6.6]. We will explain how to modify the proof of [33, Thm. 6.6] to obtain the precise statement we need.

A measured lamination μ on $\partial_0 M$ is *doubly incompressible* if there exists $c > 0$ such that if E is an essential annulus or Möbius band in M, then $i(\partial A, \mu) \geq c$. (Recall that a properly embedded annulus or Möbius band is *essential* if it is π_1-injective and can not be properly homotoped into the boundary of M.)

Theorem 3.3. *Let M be a compact, orientable, hyperbolizable 3-manifold with incompressible boundary. Suppose that $\{\rho_n\}$ is a sequence in $AH(M)$ and there exists a sequence $\{\mu_n\}$ of measured laminations on $\partial_0 M$ such that*

1. *$\{\ell_{\mu_n}(\rho_n)\}$ is bounded,*
2. *$\{\mu_n\}$ converges to a doubly incompressible measured lamination μ on $\partial_0 M$, and*
3. *the supports of $\{\mu_n\}$ Hausdorff converge to a geodesic lamination L such that every component of L contains the support of exactly one component of the support of μ.*

Then $\{\rho_n\}$ has a convergent subsequence.

Proof. We first show, given our assumptions on μ and L, that if $\pi_1(M) \times \mathcal{T} \to \mathcal{T}$ is a small, minimal action of $\pi_1(M)$ on an $\mathbb{R}$-tree $\mathcal{T}$, then some component L_0 of L is realized in $\mathcal{T}$ (see Lecuire [33] and Otal [46] for the definitions of the terminology from the theory of $\mathbb{R}$-trees used here.) Lecuire [33, Prop. 6.1] showed that, since μ is doubly incompressible, some component μ_0 of μ is realized in $\mathcal{T}$. Let S be the component of $\partial_0 M$ containing μ_0 and let $\mathcal{T}_S$ be a minimal subtree of $\mathcal{T}$ invariant under the action of $\pi_1(S) \subset \pi_1(M)$.

By Skora's Theorem [47], the action of $\pi_1(S)$ on $\mathcal{T}_S$ is dual to a measured geodesic lamination β on S. Since μ_0 is realized in $\mathcal{T}$, and hence in $\mathcal{T}_S$, μ_0 intersects β transversely. Let L_0 be the component of L containing the support of μ_0. Assumption (3) guarantees that L_0 is obtained from the support of μ_0 by adding finitely many isolated non-compact leaves. Results of Otal [46, Corol. 3.1.3 and Thm. 3.1.4] then imply that L_0 is realized in $\mathcal{T}_S$ and hence in $\mathcal{T}$.

One may now complete the proof of Theorem 3.3 exactly as in the proof of [33, Prop. 6.6]. If $\{\rho_n\}$ does not have a convergent subsequence, then there exists a subsequence converging to a small, minimal action of $\pi_1(M)$ on a $\mathbb{R}$-tree $\mathcal{T}$ (see Morgan-Shalen [40, Thm. II.4.7]). Assumption (3) implies that there exist sublaminations $\{\hat{\mu}_n\}$ of $\{\mu_n\}$ so that $\{\hat{\mu}_n\}$ converges to μ_0 and the supports of $\{\hat{\mu}_n\}$ Hausdorff converge to L_0. Lecuire's version [33, Thm. 6.5] of Otal's Continuity Theorem (see [46, Thm. 4.0.1] and [45, Thm. 3.1]), then implies, since L_0 is realized in $\mathcal{T}$, that if $\{\gamma_n\}$ is a sequence of multi-curves which Hausdorff converge to L_0 and $\sigma_0 : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ is a Fuchsian representation, then

$$\frac{\ell_{\gamma_n}(\rho_n)}{\ell_{\gamma_n}(\sigma_0)} \to \infty.$$

Since we may approximate each $\hat{\mu}_n$ arbitrarily closely by laminations supported on multicurves, $\ell_{\hat{\mu}_n}(\rho_n) \to \infty$, which contradicts Assumption (1). This contradiction completes the proof. □

It is well-known that the union of the parabolic locus and the ending laminations of a quasiconformally rigid representation is doubly incompressible, see for example the discussion in Anderson-Lecuire [4, Section 2.9]. We include a proof since we could not find a complete argument in the literature.

Lemma 3.4. *Suppose that M has incompressible boundary and that τ is a quasiconformally rigid representation in $AH_0(M)$. If μ is a measured lamination on $\partial_0 M$ whose support is*

$$\lambda_\tau = p_\tau \cup \bigcup_{W\in\mathcal{W}_\tau} \lambda_W$$

then μ is doubly incompressible.

Proof. First recall that if A is an essential annulus or Möbius band in M, then $\partial A \cap \partial_0 M$ cannot be homotoped into p_τ (since the fundamental groups of the cusps represent distinct conjugacy classes of maximal abelian subgroups). Since every simple closed curve in a pair of pants is peripheral and each ending lamination fills a component of $\partial_0 M - p_\tau$, every essential annulus or Möbius band A in M must intersect the support of μ. In particular $i(\partial A, \mu) > 0$. So, if μ is not doubly incompressible, there exists a sequence $\{A_i\}$ of distinct essential annuli or Möbius bands in M so that $\lim i(\partial A_i, \mu) = 0$.

We recall that there is a proper compact submanifold $\Sigma(M)$ of M, called the *characteristic submanifold*, so that each component of $\Sigma(M)$ is either (a) an I-bundle Σ_0 over a compact surface F_0 of negative Euler characteristic whose intersection $\partial_0\Sigma_0$ with $\partial_0 M$ is the associated ∂I-bundle or (b) a solid or thickened torus such that each component of its frontier is an essential annulus. Moreover, every essential annulus or Möbius band A in M is isotopic into Σ and if A is isotopic into a component of type (a), then it is isotopic to a *vertical* annulus or Möbius band, i.e. one which is a union of fibers of the bundle. (See Jaco-Shalen [27] or Johannson [28] for the general theory of characteristic submanifolds, and [23, Section 5] for a discussion of the theory in the restricted setting of hyperbolizable 3-manifolds.) We note that there are only finitely many isotopy classes of essential annuli or Möbius bands in a component of type (b), so we may pass to a subsequence so that each A_i is a vertical annulus or Möbius band in an interval bundle component Σ_0 of $\Sigma(M)$.

Let λ_∞ be the limit of ∂A_i in the space $PML(\partial_0\Sigma_0)$ of projective measured laminations on $\partial_0\Sigma_0$. Notice that $i(\lambda_\infty,\mu)=0$. Let G_0 be an essential subsurface of $\partial_0\Sigma_0=\Sigma_0\cup\partial M$ which contains λ_∞ so that λ_∞ fills G_0. Let $\iota_0:\Sigma_0\to\Sigma_0$ be the orientation reversing involution of Σ_0 which preserves each fiber of the bundle. Since $\partial A_i=\iota(\partial A_i)$ for all i, we may assume that G_0 is also invariant under ι_0. If ∂G_0 is non-empty, then the components of ∂G_0 bound a collection of essential annuli and Möbius bands which are disjoint from μ, which is impossible.

It remains to consider the case when $M=\Sigma_0$ and $G_0=\partial_0\Sigma_0=\partial M$. If M is untwisted, then $\nu_+(\rho)=\nu_-(\rho)$, which is impossible (see Thurston [49, Prop. 9.3.7] or Ohshika [42, Lem. 3.12]). If M is twisted, let $\widehat{M}$ be the double cover which is an untwisted interval bundle. The associated cover $\widehat{N}_\tau$ of N_τ has two geometrically infinite ends with the same ending lamination, which is again impossible. □

Next we construct a sequence of laminations whose ρ_n-length converges to zero and such that the laminations converge to a measured lamination whose support is the collection of ending invariants for τ.

Lemma 3.5. *Suppose that M has incompressible boundary, τ is a quasiconformally rigid representation in $AH_0(M)$, $\{\rho_n\}\subset \mathrm{int}(AH_0(M))$ and $\{\rho_n\}$ is eventually contained in any set of the form $\mathcal{U}(\delta,\mathbb{U},\tau)$. Then, after possibly passing to a subsequence, there exist measured laminations $\{\mu_n\}$ such that*

(1) $\{\ell_{\mu_n}(\rho_n)\}$ is bounded,
(2) $\{\mu_n\}$ converges to a measured lamination μ with support λ_τ, and
(3) the supports of $\{\mu_n\}$ Hausdorff converge to a geodesic lamination L such that every component of L contains the support of exactly one component of the support of μ.

Proof. Let S be a component of $\partial_0 M$. If $W \subset S$ is an essential subsurface, $\rho \in AH(S)$, and $L > 0$, let $\mathcal{C}_W(\rho, L)$ denote the set of curves $\gamma \in \mathcal{C}(W)$ whose length $\ell_\gamma(\rho) \leq L$.

Theorem 1.2 in [14] gives control of the set $\mathcal{C}_W(\rho, L)$ when L is large enough. Specifically, there exists $L_S > 0$ so that if $L \geq L_S$ then there exists $D = D(L)$ such that, if $\mathcal{C}_W(\rho, L)$ is nonempty then the Hausdorff distance from a geodesic connecting $\pi_W(\nu_+(\rho))$ to $\pi_W(\nu_-(\rho))$ to $\mathcal{C}(W, L)$ is at most D. In particular, if $\pi_W(\nu_+(\rho)) \in \mathcal{C}(W)$, then there exists a curve in $\mathcal{C}_W(\rho, L)$ within distance D of $\pi_W(\nu_+(\rho))$.

To apply this to our sequence, let W be a component of $\mathcal{W}_\tau$. For each n, let g_n^W be a hyperbolic metric with geodesic boundary on W and let $f_n^W : (W, g_n^W) \to N_{\rho_n}$ be a 1-Lipschitz map in the homotopy class of $\rho_n|_{\pi_1(W)}$, e.g. a pleated surface. Since $\ell_{\partial W}(\rho_n) \to 0$, there exists $L_W > L_S$, so that, for all large enough n, g_n^W contains a non-peripheral curve of length at most L_W, so $\mathcal{C}_W(\rho_n, L_W)$ is nonempty. Applying the previous paragraph to $\rho_n^S = \rho_n|_{\pi_1(S)}$, we obtain a curve c_W^n in W with $d_{\mathcal{C}(W)}(c_W^n, \pi_W(\nu(\rho_n))) \leq D_W$ and $\ell_{c_W^n}(\rho_n) \leq L_W$, for some uniform $D_W = D(L_W)$.

Since $\{\pi_W(\nu(\rho_n))\}$ converges to $\lambda_W \in \partial_\infty \mathcal{C}(W)$, we may choose a sequence $r_n^W \to \infty$ so that any subsequence of $\{c_n^W / r_n^W\} \subset \mathcal{ML}(W)$ has a subsequence which converges to a measured lamination with support λ_W, where $\mathcal{ML}(W)$ is the space of measured laminations on W, see Klarreich [30, Thm. 1.4]. Moreover, any subsequence of $\{c_n^W\}$ has a subsequence which Hausdorff converges to a connected geodesic lamination supported in the interior of W. If

$$\mu_n = p_\tau \cup \bigcup_{W \in \mathcal{W}_\tau} \frac{c_n^W}{r_n^W}$$

then $\lim \ell_{\mu_n}(\rho_n) = 0$ and every subsequence of $\{\mu_n\}$ has a subsequence converging to a measured lamination whose support is λ_τ so that the supports Hausdorff converge to a geodesic lamination satisfying assumption (3). □

We now combine the preceding results and topological arguments to establish Proposition 3.1.

Proof of Proposition 3.1. It remains to prove that if $\{\rho_n\}$ is eventually contained in every set of the form $\mathcal{U}(\delta, \mathbb{U}, \tau)$, then $\{\rho_n\}$ converges to τ (or accumulates on $\{\tau, \bar{\tau}\}$ when M is an interval bundle and $\mathcal{W}_\tau$ is empty). It suffices to prove that every such sequence has a subsequence that converges to τ (or into $\{\tau, \bar{\tau}\}$ if M is an interval bundle and $\mathcal{W}_\tau$ is empty).

If $\{\rho_n\}$ is eventually contained in every set of the form $\mathcal{U}(\delta, \mathbb{U}, \tau)$, Lemmas 3.4 and 3.5 imply that, after possibly passing to a subsequence, there exist measured laminations $\{\mu_n\}$ so that $\ell_{\mu_n}(\rho_n) \to 0$, $\{\mu_n\}$ converges to a doubly incompressible measured lamination μ with support λ_τ, and the supports of $\{\mu_n\}$ converge to a measured lamination L with the property that every component of L contains exactly one component of the support of μ. Theorem 3.3 then implies that, passing to a further subsequence, $\{\rho_n\}$ converges to some $\rho \in AH(M)$.

We must show that $\rho = \tau$ (or that $\rho \in \{\tau, \bar{\tau}\}$ if M is an interval bundle and $\mathcal{W}_\tau$ is empty). To do so, we will show that N_ρ has the same marked homeomorphism type as N_τ and that both manifolds have the same end-invariants, and then apply the Ending Lamination Theorem [16].

Let M_ρ be a relative compact core for N^0_ρ and let $P_\rho = \partial M_\rho \cap \partial N^0_\rho$. Similarly, let M_τ be a relative compact core for N^0_τ and let $P_\tau = \partial M_\tau \cap \partial N^0_\tau$. We claim that there exists a homotopy-equivalence $j : M_\tau \to M_\rho$, in the homotopy class of $\rho \circ \tau^{-1}$, such that

(1) j takes P_τ homeomorphically to a subcollection $\widehat{P}$ of P_ρ,
(2) there exists a submanifold Z of $\partial M_\tau \setminus P_\tau$ which consists of a compact core of each component of $\partial M_\tau \setminus P_\tau$ which is not a thrice-punctured sphere, such that j restricts to an orientation-preserving embedding of Z into $\partial M_\rho \setminus \widehat{P}$ and each component of $j(Z)$ is a compact core for a component of $\partial M_\rho \setminus P_\rho$.

In this situation, Proposition 8.1 of Canary-Hersonsky [22] asserts that j is homotopic, as a map of pairs, to a homeomorphism of pairs

$$J : (M_\tau, P_\tau) \to (M_\rho, \widehat{P})$$

which agrees with j on Z. (Canary and Hersonsky's result is a nearly immediate consequence of Johannson's Classification Theorem [28]. In our setting, each component of $P_\tau \cup Z$ is incompressible and there are no essential annuli in M with boundary in $\partial M - (P_\tau \cup Z)$ and Johannson's Classification Theorem may be viewed as a relative version of his theorem that any homotopy equivalence between acylindrical 3-manifolds is homotopic to a homotopy equivalence.) Furthermore, we will see that every component of $j(Z)$ is a compact core for a component of $M_\rho - \widehat{P}$ which bounds a geometrically infinite end of N^0_ρ. It follows that $\widehat{P} = P_\rho$.

Since M_τ and M_ρ are aspherical and $\rho \circ \tau^{-1}$ gives an isomorphism of $\pi_1(M_\tau) = \tau(\pi_1(M))$ to $\pi_1(M_\rho) = \rho(\pi_1(M))$, there is a homotopy equivalence $j : M_\tau \to M_\rho$ in the homotopy class of $\rho \circ \tau^{-1}$. To establish part (1), note that $\ell_\rho(p_\tau) = 0$. Hence we may choose j to take all annular components of P_τ to annular components of P_ρ. Moreover, if T is a toroidal component of P_τ, then $j(T)$ is homotopic to a toroidal component of P_ρ, so we may assume that $j(T) \subset P_\rho$ in this case as well. Since the restriction of j to any component of P_τ is a homotopy equivalence to the image component of P_ρ, we may assume that the restriction of j to each component of P_τ is a homeomorphism onto a component of P_ρ.

For part (2), we will show that geometrically infinite ends of N^0_τ are associated to geometrically infinite ends of N^0_ρ with the same ending lamination. If $W \in \mathcal{W}_\tau$ and S is the component of $\partial_0 M$ containing W, then $\{\pi_W(\nu(\rho_n))\}$

converges to λ_W, so Theorem 2.5 implies that $\rho^S = \rho|_{\pi_1(S)}$ has an upward-pointing geometrically infinite end with support W and ending lamination λ_W. Part (2) then follows from the following more general claim.

Lemma 3.6. *Suppose that M is a compact, orientable, hyperbolizable 3-manifold with non-empty incompressible boundary and that W is an essential subsurface of a boundary component S of M which is not an annulus or a pair of pants. If $\rho \in AH(M)$, M_ρ is a relative compact core for N^0_ρ and ρ^S has an outward-pointing geometrically infinite end with base surface W, then there exists a homotopy equivalence $g : M \to M_\rho$ which restricts to an orientation-preserving homeomorphism of W to a subsurface $g(W)$ in ∂M_ρ which bounds a geometrically infinite end of N^0_ρ*

Proof. The Covering Theorem ([20] and [49, Thm. 9.2.2]) implies that the covering map $\pi : N_{\rho^S} \to N_\rho$ restricts to a finite-to-one cover from a neighborhood U of the end of $N^0_{\rho^S}$ associated to W to a neighborhood $\widehat{U}$ of a geometrically infinite end of N^0_ρ. Let W' be the component of $\partial M_\rho \setminus P_\rho$ associated to the end of N^0_ρ with neighborhood $\widehat{U}$. Since U has a subneighborhood homeomorphic to $W \times (0, \infty)$ which maps into a subneighborhood of $\widehat{U}$ which is homeomorphic to $W' \times (0, \infty)$, we see that there exists a homotopy equivalence $g : M \to M_\rho$ so that $g|_W$ is a proper π_1-injective orientation-preserving map onto W', so we may assume $g|_W$ is an orientation-preserving finite cover of its image (see [28, Prop. 3.3.]).

If $g|_W$ is not a homeomorphism, then there exists a curve α on W which is indivisible in $\pi_1(W)$, so that $g(\alpha)$ is divisible in $\pi_1(W')$. It follows that α is a peripheral curve which is divisible in M, but not in ∂M. A result of Johannson [28, Lem. 32.1] implies that either (i) α bounds an immersed essential Möbius band B in an interval bundle component Σ_0 (with base surface F_0) of $\Sigma(M_\tau)$, or (ii) α is homotopic to the core curve of an annulus in the frontier of a solid torus component V of $\Sigma(M_\tau)$. In case (i), Johannson's Classification Theorem [28, Thm. 24.2] implies that g is homotopic to a homotopy equivalence which restricts to a homeomorphism of Σ_0 to an interval bundle component $g(\Sigma_0)$ of $\Sigma(M_\rho)$, so $g(\alpha)$ bounds an immersed essential Möbius band in M_ρ. In case (ii), Johannson's Classification Theorem implies that g is homotopic to a homotopy equivalence which takes V to a solid torus component $g(V)$ of $\Sigma(M_\rho)$ and restricts to a homeomorphism between the frontier of V in M and the frontier of $g(V)$ in M_ρ, so $g(\alpha)$ is is homotopic to the core curve of an annulus in the frontier of $g(V)$. In either case, $g(\alpha)$ cannot be divisible in $\pi_1(W') \subset \pi_1(M_\rho)$, so we have again achieved a contradiction. □

Since we have adjusted j to satisfy conditions (1) and (2) above, we may apply Proposition 8.1 of [22] to upgrade j to a homeomorphism of pairs $J : (M_\tau, P_\tau) \to (M_\rho, P_\rho)$.

If J is orientation-preserving, then the Ending Lamination Theorem tells us that $\rho = \tau$ in $AH(M)$ and we are done. In particular, we are done if

$\mathcal{W}_\tau$ is non-empty, since the restriction of J to each element of $\mathcal{W}_\tau$ is an orientation-preserving homeomorphism.

If J is orientation-reversing, then the Ending Lamination Theorem tells us that $\rho = \bar{\tau}$ in $AH(M)$. Since we may take $M_{\bar{\tau}}$ to be M_τ with the opposite orientation, J is an orientation-reversing involution of M_τ which is homotopic to the identity. It follows from the lemma below that M_τ, and therefore M, is an interval bundle. So, since M is an interval bundle, $\mathcal{W}_\tau$ is empty, and ρ lies in $\{\tau, \bar{\tau}\}$, we are again done.

Lemma 3.7. *If M is a compact, orientable, hyperbolizable 3-manifold with non-empty incompressible boundary, and $g : M \to M$ is an orientation-reversing involution which is homotopic to the identity, then M is an interval bundle.*

Proof. Let S be a boundary component of M and let $H : S \times [0,1] \to M$ be the restriction to S of the homotopy of g to the identity map. Notice that H is not properly homotopic to a map $\hat{H}$ with image in ∂M, since then $\hat{H}$ would be a homotopy of an orientation-reversing involution of S to an orientation-preserving homeomorphism of S. Then, a generalization of Waldhausen's theorem, see Hempel [25, Theorem 13.6], implies that H is properly homotopic to a covering map. Therefore, M must be an interval bundle, see Hempel [25, Thm 10.6]. □

□

4. Bounds on the skinning map

Thurston's skinning map records the geometry, i.e. end invariants, of the "inward-pointing" ends of the covers of a hyperbolic 3-manifold associated to its peripheral subgroups. In this section, we use the results of [12] to show that certain aspects of the geometry of these inward-pointing ends is uniformly bounded as one approaches a quasiconformally rigid hyperbolic 3-manifold.

Let M be a compact, orientable hyperbolizable 3-manifold with incompressible boundary, which is not an untwisted interval bundle. If $\rho \in AH_0(M)$ and S is a component of $\partial_0 M$, then $\rho^S = \rho|_{\pi_1(S)}$ is a Kleinian surface group with end invariants $(\nu_+(\rho^S), \nu_-(\rho^S))$. We assume that we have chosen an orientation on S which is consistent with the orientation on M, so that $\nu_+(\rho^S)$ is the restriction $\nu(\rho)|_S$ of $\nu(\rho)$ to S. One then defines a map σ_S, with image in the set $\mathcal{E}(S)$ of ending invariants on S, by setting $\sigma_S(\rho) = \nu_-(\rho^S)$. The *skinning map* is the product map

$$\sigma_M = \prod \sigma_S : AH_0(M) \to \mathcal{E}(\partial_0 M)$$

where the product is taken over all components of $\partial_0 M$ and $\mathcal{E}(\partial_0 M)$ is the product of the $\mathcal{E}(S)$.

Remark: In this paper we only actually apply the skinning map to representations in $\mathrm{int}(AH_0(M))$. If ∂M does not contain tori, then the image of

$\rho \in \mathrm{int}(AH_0(M))$ under σ_M will lie in the Teichmüller space $\mathcal{T}(\partial M)$. If ∂M does contain tori, let t be the collection of curves on $\partial_0 M$ which are homotopic into a toroidal component of ∂M. If $\rho \in \mathrm{int}(AH_0(M))$, then $\sigma_M(\rho)$ consists of the parabolic locus t and a conformal structure on $\partial_0 M - t$. (See [15, Sec. 2] for a more detailed discussion of the skinning map in the setting of 3-manifolds with incompressible boundary.)

Proposition 4.1. *Let M be a compact, orientable, hyperbolizable 3-manifold with incompressible boundary and let μ be a complete marking on $\partial_0 M$. If $\tau \in AH_0(M)$ is quasiconformally rigid, then there exists a neighborhood $\mathcal{U}$ of τ such that for all $\rho \in \mathcal{U}$ and $\alpha \in p_\tau$,*

$$\mathbf{m}_\alpha(\sigma_M(\rho), \mu) < R.$$

Proof. If the proposition fails, then there exists a sequence $\{\rho_n\}$ in $\mathrm{int}(AH_0(M))$ converging to τ and $\alpha \in p_\tau$ such that $\mathbf{m}_\alpha(\sigma_M(\rho_n), \mu) \to \infty$. Let S be the component of $\partial_0 M$ containing α, so $\mathbf{m}_\alpha(\sigma_M(\rho_n), \mu) = \mathbf{m}_\alpha(\sigma_S(\rho_n), \mu)$ and $\{\rho_n^S\}$ converges to $\tau^S = \tau|_{\pi_1(S)}$.

In order to apply Theorem 2.4, we need to find a geometric limit $\hat\Gamma^S$ for $\{\rho_n(\pi_1(S))\}$ so that a compact core of N_{τ^S} embeds in $\widehat{N}^S = \mathbb{H}^3/\hat\Gamma_S$. Pass first to a subsequence so that $\{\rho_n(\pi_1(M))\}$ converges geometrically to $\hat\Gamma$ and, by passing to a further subsequence, so that $\{\rho_n(\pi_1(S))\}$ converges geometrically to $\hat\Gamma^S \subset \hat\Gamma$. It follows from Proposition 2.2 that there exists a compact core C_τ for N_τ which embeds, under the natural covering map, in $\widehat{N} = \mathbb{H}^3/\hat\Gamma$.

Since $\tau \in AH_0(M)$, there exists a homeomorphism $h_\tau : M \to M_\tau$ so that $(h_\tau)_*$ is conjugate to τ, where M_τ is a relative compact core for N_τ^0. Moreover, by a result of McCullough, Miller and Swarup [37], there exists a homeomorphism $g : M_\tau \to C_\tau$ so that $f = g \circ h_\tau$ is in the homotopy class of τ. Since $f(S)$ embeds in $\hat N$, it admits a compact regular neighborhood R which embeds in $\widehat{N}$. Then, R lifts to a compact core for N_{τ^S} which embeds in $\widehat{N}^S = \mathbb{H}^3/\hat\Gamma^S$.

Notice that $\sigma_S(\rho_n) = \nu(\rho_n^S)_-$ for all n. Since α is an upward-pointing parabolic for $\tau|_{\pi_1(S)}$, Theorem 2.4 then implies that if μ is a complete marking on S, then $\{\mathbf{m}_\alpha(\sigma_S(\rho_n), \mu)\}$ is eventually bounded, which provides a contradiction. ☐

5. Proofs of main results

Suppose that τ is quasiconformally rigid. We may assume, by precomposing by an element of $\mathrm{Out}(\pi_1(M))$ and by replacing M by M_τ, that $\tau \in AH_0(M)$. Theorem 2.1 then implies that τ lies in the boundary of $\mathrm{int}(AH_0(M))$ and does not lie in the boundary of any other component of the interior of $AH(M)$. Therefore, Theorem 1.2, which asserts that no component of $\mathrm{int}(AH(M))$ self-bumps at τ, follows immediately from the following result.

Proposition 5.1. *Let τ be a quasiconformally rigid representation in $AH_0(M)$. Given any open neighborhood V of τ, there exists a sub-neighborhood $\hat{V} \subset V$ of τ so that any two points in $\hat{V} \cap \mathrm{int}(AH_0(M))$ can be joined by a path in $V \cap \mathrm{int}(AH_0(M))$.*

Notice that Theorem 1.1, which asserts that $AH(M)$ is locally connected at τ, follows from the facts that no two components of $\mathrm{int}(AH(M))$ bump at τ (Theorem 2.1), no component of $\mathrm{int}(AH(M))$ self-bumps at τ, and that $AH(M)$ is the closure of its interior [16].

Our main tool in constructing the required paths is the following result from [13] which allows us to pinch curves in the conformal boundary while controlling complementary subsurface projections.

Lemma 5.2. ([13, Lemma 6.1]) *Given a (possibly disconnected) surface S and constants K and $\delta > 0$, there exists $c = c(S)$ and $h = h(\delta, K, S)$ such that if $X \in \mathcal{T}(S)$, μ is a complete marking of S and $\boldsymbol{\alpha}$ is a curve system on S such that*

$$\mathbf{m}_\alpha(X, \mu) > h(\delta, K, S)$$

for each component α of $\boldsymbol{\alpha}$, then there exists a path $\{X_t : t \in [0,1]\}$ in $\mathcal{T}(S)$ with $X_0 = X$ such that

1. *$l_\alpha(X_1) < \delta/2$ for each $\alpha \in \boldsymbol{\alpha}$,*
2. *$\mathbf{m}_\alpha(X_t, \mu) > K$ for each $\alpha \in \boldsymbol{\alpha}$ and each $t \in [0,1]$, and*
3. *$\mathrm{diam}(\pi_W(\{X_t : t \in [0,1]\})) < c$, for any subsurface W disjoint from $\boldsymbol{\alpha}$.*

Remark: In Lemma 6.1 in [13] the basepoint μ is a conformal structure on S, while here it is a complete marking. However, as we are only interested in the projection to the curve complex and for any conformal structure there is a complete marking that has coarsely the same image in the projection to any curve complex the statements are equivalent. We are using a complete marking here to match with Proposition 4.1.

Proof of Proposition 5.1. Let V' be an open neighborhood of $\bar{\tau}$ in $AH(M)$ so that V and V' have disjoint closures. If $\tau = \bar{\tau}$, let $V = V'$. (Note, we only need V' in the case of interval bundles). Proposition 3.1 implies that there exists $\delta > 0$ and $\mathbb{U} = \{U_W\}_{W \in \mathcal{W}_\tau}$ so that each U_W is a neighborhood of $\lambda_W \in \partial_\infty \mathcal{C}(W)$ and

$$\mathcal{U}(\delta, \mathbb{U}, \tau) \subset V \cup V'.$$

Let $\rho, \hat{\rho} \in \mathcal{U}(\delta, \mathbb{U}, \tau)$. If all the curves in p_τ are short in the conformal boundary of the hyperbolic manifolds N_ρ and $N_{\hat{\rho}}$, then it is not hard to construct a path between ρ and $\hat{\rho}$ that stays in the given neighborhood. The central difficulty in the general case is that while each curve $\alpha \in p_\tau$ is short in the hyperbolic 3-manifold, these curves may not be short in the conformal boundary.

More formally, Lemma 5.11 of [13] gives a collection $\mathbb{U}' = \{U'_W\}_{W \in \mathcal{W}_\tau}$ of sub-neighborhoods of $\mathbb{U}$ such that if $\rho, \hat{\rho} \in \mathrm{int}(AH_0(M))$ satisfy

(A) $\ell_\alpha(\nu(\rho)), \ell_\alpha(\nu(\hat\rho)) < \delta/2$ for all $\alpha \in p_\tau$, and
(B) $\pi_W(\rho), \pi_W(\hat\rho) \in U'_W$ for all $W \in \mathcal{W}_\tau$,

then ρ and $\hat\rho$ can be connected by a path $\{\rho_t \mid t \in [0,1]\}$ so that $\ell_\alpha(\nu(\rho_t)) < \delta/2$ and $\pi_W(\rho_t) \in U_W$ for all $t \in [0,1]$ and all $W \in \mathcal{W}_\tau$. Bers [5, Thm. 3] proved that if α is a simple closed curve on $\partial_0 M$, and M has incompressible boundary, then $\ell_\alpha(\rho) \le 2\ell_\alpha(\nu(\rho))$, so the path is entirely contained in $\mathcal{U}(\delta, \mathbb{U}, \tau)$.

Thus we will complete the proof by finding $\epsilon > 0$ and a collection $\widehat{\mathbb{U}} = \{\hat U_W\}_{W\in\mathcal{W}_\tau}$ of sub-neighborhoods of $\mathbb{U}$, so that if $\rho \in \mathcal{U}(\epsilon, \widehat{\mathbb{U}}, \tau)$, then ρ can be connected to a representation satisfying (A) and (B) by a path that is entirely contained in $\mathcal{U}(\delta, \mathbb{U}, \tau)$. It then follows that any two points in $\mathcal{U}(\epsilon, \widehat{\mathbb{U}}, \tau)$ can be joined by a path in $\mathcal{U}(\delta, \mathbb{U}, \tau)$.

Proposition 3.1 will then imply that there exists a neighborhood $\hat V \subset V$ of τ so that $\hat V \cap \mathrm{int}(AH_0(M))$ is contained in $\mathcal{U}(\epsilon, \widehat{\mathbb{U}}, \tau)$. It follows that any two points in $\hat V \cap \mathrm{int}(AH_0(M))$ can be joined by a path in $\mathcal{U}(\delta, \mathbb{U}, \tau) \subset V \cup V'$. Since V and V' have disjoint closures, the path must be entirely contained in V.

We fix a complete marking μ on $\partial_0 M$. Proposition 4.1 provides a neighborhood $\mathcal{U}(\delta_0, \mathbb{U}_0, \tau)$ of τ, where $\mathbb{U}_0 = \{(U_0)_W\}_{W\in\mathcal{W}_\tau}$, and $R > 0$, such that

$$\mathbf{m}_\alpha(\sigma_M(\rho), \mu) < R.$$

if $\rho \in \mathcal{U}(\delta_0, \mathbb{U}_0, \tau)$. We may assume, without loss of generality, that $\delta < \delta_0$ and $\mathbb{U} \subset \mathbb{U}_0$.

We will apply Lemma 5.2 to find a small enough neighborhood of τ so that any representation in the neighborhood may be joined to a representation satisfying (A) and (B) by a path $\{\rho_t\}$ in $\mathcal{U}(\delta, \mathbb{U}, \tau)$. It is easy to use part (3) of Lemma 5.2 to choose the sub-neighborhoods $\widehat{\mathbb{U}}$ so that the projection of the path stays in $\mathbb{U}$. It is more difficult to ensure that each curve α in p_τ has length less than δ on the entire path. Here we use part (2) of Lemma 5.2 to show that $\mathbf{m}_\alpha(\nu(\rho_t), \nu(\sigma_M(\rho_t)))$ is large, which in turn, by Theorem 2.3, will imply that α is short.

Theorem 2.3 provides $K > 0$, so that if γ is a simple closed curve in $\partial_0 M$, $\rho \in AH_0(M)$ and $\mathbf{m}_\gamma(\nu(\rho), \sigma_M(\rho)) > K$, then

$$\ell_\gamma(\rho) < \delta/2.$$

Let $c = c(\partial_0 M)$ and $h = h(\delta, K + R, \partial_0 M)$ be the constants given by Lemma 5.2. Theorem 2.3 implies that there exists $\epsilon > 0$ so that if γ is a simple closed curve in $\partial_0 M$ and $\ell_\gamma(\rho) < \epsilon$, then

$$\mathbf{m}_\gamma(\nu(\rho), \sigma_M(\rho)) > h + R.$$

Let $\widehat{\mathbb{U}} = \{\hat U_W\}_{W\in\mathcal{W}_\tau}$ be a collection of neighborhoods of the ending laminations of τ so that, for each $W \in W_\tau$, the neighborhood of $\hat U_W$ of radius c in $\mathcal{C}(W)$ is contained in U'_W.

If $\rho \in \mathcal{U}(\epsilon, \widehat{\mathbb{U}}, \tau)$, then $\ell_\alpha(\rho) < \epsilon$, for all $\alpha \in p_\tau$, so

$$\mathbf{m}_\gamma(\nu(\rho), \sigma_M(\rho)) > h + R,$$

which implies that $\mathbf{m}_\gamma(\nu(\rho), \mu) > h$, since $\mathbf{m}_\gamma(\mu, \sigma_M(\rho)) < R$. Lemma 5.2 then implies that there exists a path $\{X_t \mid t \in [0,1]\}$ in $\mathcal{T}(\partial_0 M)$ with $X_0 = \nu(\rho)$ so that

(1) $l_\alpha(X_1) < \delta/2$ for each $\alpha \in p_\tau$,
(2) $\mathbf{m}_\alpha(X_t, \mu) > K + R$ for each $\alpha \in p_\tau$ and each $t \in [0,1]$, and
(3) $\operatorname{diam}(\pi_W(\{X_t : t \in [0,1]\})) < c$ for all $W \in \mathcal{W}_\tau$.

Let $\{\rho_t \mid t \in [0,1]\}$ be the path in $\operatorname{int}(AH_0(M))$ with $\nu(\rho_t) = X_t$ for all t. It only remains to check that $\{\rho_t\} \subset \mathcal{U}(\delta, \mathbb{U}, \tau)$ for all t and that ρ_1 satisfies properties (A) and (B).

Property (1) implies that $\ell_\alpha(\nu(\rho_1)) < \delta/2$ for all $\alpha \in p_\tau$, so ρ_1 satisfies (A). Property (3) implies that if $W \in \mathcal{W}_\tau$, then

$$\operatorname{diam}(\pi_W(\{X_t : t \in [0,1]\})) < c.$$

It follows from the definition of $\hat{U}_W$ and the fact that $\pi_W(X_0) = \pi_W(\rho) \in \hat{U}_W$, that

$$\pi_W(\rho_t) = \pi_W(X_t) \in U'_W \subset U_W$$

for all $t \in [0,1]$ and all $W \in \mathcal{W}_\tau$. In particular, ρ_1 also satisfies (B).

In order to verify that $\rho_t \in \mathcal{U}(\delta, \mathbb{U}, \tau)$ for all $t \in [0,1]$, it remains to check that $\ell_\alpha(\rho_t) < \delta$ for all $t \in [0,1]$. If this is not the case, there exists $s \in [0,1]$ and $\alpha \in p_\tau$ so that $\ell_\alpha(\rho_s) = \delta < \delta_0$. Since $\rho_s \in \mathcal{U}(\delta_0, \mathbb{U}_0, \tau)$, we know that $\mathbf{m}_\alpha(\sigma_M(\rho_s), \mu) < R$. Therefore, by applying property (2) above and the triangle inequality for $\mathbf{m}_\alpha$, we see that

$$\begin{aligned}
\mathbf{m}_\alpha(\nu(\rho_s), \sigma_M(\rho_s)) &= \mathbf{m}_\alpha(X_s, \sigma_M(\rho_s)) \\
&\geq \mathbf{m}_\alpha(X_s, \mu) - \mathbf{m}_\alpha(\sigma_M(\rho_s), \mu) \\
&> K + R - R = K.
\end{aligned}$$

So, by our assumptions on K, $\ell_\alpha(\rho_s) < \delta/2$, which is a contradiction.

This completes the proof that $\rho_t \in \mathcal{U}(\delta, \mathbb{U}, \tau)$ for all $t \in [0,1]$, and hence Proposition 5.1. □

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