Bending map
and strong convergence

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Abstract

The bending map of a hyperbolic 3-manifold with boundary maps a geometrically hyperbolic metric to its bending measured geodesic lamination. We show that the bending map is proper. Combining this we the result of [Le3], we deduce a criterion for the strong convergence of a sequence of geometrically finite representations to a geometrically representation. Using the properness of the bending map, we define a subset of $\mathscr{ML}(\partial M)$ on which the action of the group of isotopy classes of homeomorphisms of M is properly discontinuous.

1 Introduction

Consider a compact, orientable 3-manifold M that is hyperbolic, namely the interior of M is endowed with a complete metric σ of constant sectional curvature -1. Assume that ∂M contains a surface of genus > 1. A fundamental subset of (M, σ) is its convex core $N(\sigma)$, defined as the smallest non-empty closed subset of the interior of M that is locally convex and homotopy equivalent to M. Its boundary $\partial N(\sigma)$ is a pleated surface, namely it is almost everywhere totally geodesic and it is bent along a geodesic lamination. The amount of bending is described by a measured geodesic lamination called the bending measured geodesic lamination of σ (cf. [Th1] or [CEG]). This yields a bending map $b : \mathscr{H}(M) \to \mathscr{ML}(\partial M)$ which to a complete hyperbolic metric associates its bending measured geodesic lamination. In the present paper, we investigate some properness properties of this bending map.

We will consider isotopy classes of hyperbolic metric on the interior of M. Two metrics σ_1 and σ_2 are identified if there exists a diffeomorphism $f : M \to M$ isotopic to the identity such that $\sigma_1 = f^*\sigma_2$. We will topologize the set $\mathscr{H}(M)$ of isotopy classes of complete hyperbolic metrics in the following way. Let us choose a point x in $int(M)$. A metric σ_2 lies in a (k,r)-neighborhood of σ_1 if there exists a diffeomorphism $g : M \to M$ isotopic to the identity such that the restriction of g to $B(x,r) \subset (M, \sigma_1)$ is a k-quasi-isometry into its image in (M, σ_2). We obtain a basis of neighborhoods of σ_1 by letting k tend to 1 and r tend to $+\infty$. The topology defined in this way does not depend on the choice of the point x.

In the present paper we will only consider *geometrically finite* hyperbolic metric on $int(M)$: a complete hyperbolic metric σ on $int(M)$ is said to be *geometrically finite* if $N(\sigma)$ has finite volume. The metric σ is *fuchsian* if $int(N(\sigma)) = \emptyset$. We will denote by $\mathscr{GF}(M) \subset \mathscr{H}(M)$ the set of isotopy classes of geometrically finite non-fuchsian metrics on $int(M)$. We will denote by $\mathscr{CC}(M) \subset \mathscr{GF}(M)$ the set of isotopy classes of non-fuchsian geometrically finite hyperbolic metrics on $int(M)$ whose only cusps are rank 2 cusps. On $\mathscr{CC}(M)$ the topology defined above is equivalent to the topology obtained when we consider quasi-isometric deformations of the whole metric on $int(M)$. Let us denote by $b_{\mathscr{CC}}$, $b_{\mathscr{GF}}$ the restrictions of b to $\mathscr{CC}(M)$ and to $\mathscr{GF}(M)$ respectively. The continuity and the differentiability of $b_{\mathscr{CC}}$ have been studied in [KeS] and [Bo2]. Its injectivity has been established in [BoO] in the special case of weighted multi-curves. The image of $b_{\mathscr{GF}}$ has been described in [BoO] and [Le1], it is the set $\mathscr{P}(M)$ of measured geodesic laminations satisfying the following conditions:

- a) no closed leaf of λ has a weight greater than π;

- *b)* $\exists \eta > 0$ such that, for any essential annulus E, $i(\partial E, \lambda) \geq \eta$;

- c) $i(\lambda, \partial D) > 2\pi$ for any essential disc D.

In the present paper we will study properness properties of $b_{\mathscr{G}\mathscr{F}}$. As we will see, such properties are useful to establish some convergence criteria for geometrically finite metrics involving only bending measured geodesic laminations. The first result we will present is a generalization of the "lemme de fermeture" in [BoO]:

Theorem 1.1. *Let M be a compact connected orientable hyperbolic 3-manifold with boundary and let σ_n be a sequence of geometrically finite non-fuchsian metrics on the interior of M. Denote by $\lambda_n \subset \mathscr{ML}(\partial M)$ the bending measured geodesic lamination of σ_n. Suppose that λ_n converge in $\mathscr{ML}(\partial M)$ to a measured geodesic lamination λ satisfying the following conditions:*

- *b) $\exists \eta > 0$ such that, for any essential annulus E, $i(\partial E, \lambda) \geq \eta$;*

- *c) $i(\lambda, \partial D) > 2\pi$ for any essential disc D;*

- *d) there is no homeomorphism between M and an interval bundle over a pair of pants P such that the support of λ is the image of a section of the bundle over ∂P.*

Then the sequence (σ_n) contains a subsequence converging to a geometrically finite non-fuchsian metric σ_∞ on the interior of M.

Furthermore it follows from [Le3] that the bending measured geodesic lamination λ' of σ_∞ is obtained from λ by replacing by π the weights of the closed leaves which have a weight greater than π.

If we consider only convex cocompact metrics, we deduce from this theorem that $b_{\mathscr{C}\mathscr{C}}$ is proper:

Corollary. *The map $b_{\mathscr{C}\mathscr{C}}$ from $\mathscr{C}\mathscr{C}(M)$ to $b(\mathscr{C}\mathscr{C}(M))$ endowed with the weak* topology is proper.*

Remark. Recall that $\mathscr{C}\mathscr{C}(M)$ does not contain any fuchsian metric. If M admits some fuchsian metric, it is homeomorphic to an I-bundle over a compact surface S. Any hyperbolic metric on S yields a geometrically finite metric on $int(M)$ with the following bending geodesic measured lamination λ : the support of λ is a section of the I-bundle over ∂S and each leaf has a weight equal to π. When S is not a pair of pants, the set of hyperbolic metric on S is not compact. Let us denote by $\mathscr{F}(M)$ the set of fuchsian metrics on M. This set $\mathscr{F}$ is not empty if M is a handlebody or an I-bundle over a closed surface. If $\mathscr{F}(M)$ is not empty, then it is not compact. Therefore the restriction of b to $\mathscr{F}(M)$ is not proper. For a discussion about non-fuchsian metrics tending to fuchsian ones, the reader should refer to [Se]. ◇

When we deal with geometrically finite metrics, difficulties appear with leaves which have a weight greater than π. Especially, it follows from Theorem 1.1 that the bending map is discontinuous in the neighborhood of any geometrically finite metric having at least one rank 1 cusp. To overcome this difficulty, we quotient the set $\mathscr{P}^+(M)$ of measured geodesic laminations satisfying condition *b*), *c*) and *d*) by the following relation:

Let $\lambda, \mu \in \mathscr{ML}(\partial M)$ be two measured geodesic laminations and let us denote by λ' (resp. μ') the measured geodesic laminations obtained by replacing by π the weights of the leaves of λ (resp. μ) which have a weight greater than π. We set $\lambda \mathscr{R} \mu$ if and only if $\lambda' = \mu'$. We denote by

$\dot{\lambda}$ the class of λ modulo $\mathscr{R}$.

Endow $\mathscr{ML}(\partial M)$ with the weak* topology and $\mathscr{ML}(\partial M)/\mathscr{R}$ with the quotient topology. It is not hard to see that this space is a Hausdorff space (cf. [Le3, Lemma 2.3]). From $b_{\mathscr{GF}}(M)$ we obtain a quotient map $b_{\mathscr{R}} : \mathscr{GF}(M) \to \mathscr{P}^+(M)/\mathscr{R}$. We can then give a general version of Theorem 1.1 :

Theorem 1.2. *The map $b_{\mathscr{R}}$ from $\mathscr{GF}(M)$ to $\mathscr{P}^+(M)/\mathscr{R}$ is proper.*

Rather than considering isotopy classes of metrics, we could consider homotopy classes of metrics. Namely we quotient the set $\mathscr{GF}(M)$ by the group $Mod_0(M)$ of diffeomorphisms of M which are homotopic to the identity. The interesting point is that this quotient $GF(M) = \mathscr{GF}(M)/Mod_0(M)$ is the same set as the set of conjugacy classes of geometrically finite representations $\rho : \pi_1(M) \to Isom(\mathbb{H}^3)$ such that $\mathbb{H}^3/\rho(\pi_1(M))$ is homeomorphic to the interior of M. We endow this set $GF(M)$ with the quotient topology of the topology of $\mathscr{GF}(M)$. When we regard $GF(M)$ as a set of representations, the topology thus defined is the topology of the strong convergence. We have to notice that if a representation $\rho \in GF(M)$ is given (up to conjugacy) then its bending measured geodesic lamination is defined up to the action of the group $Mod_0(M)$. Let us quotient the space $\mathscr{ML}(\partial M)$ by $\mathscr{R}$ and by the action of $Mod_0(M)$: two measured geodesic laminations α and λ lie in the same class modulo $Mod_0(M) * \mathscr{R}$ if and only if there is a diffeomorphism $\phi : M \to M$ homotopic to the identity such that $\lambda' = \phi(\alpha')$ where λ' and α' are the same as in the definition of $\mathscr{R}$ (namely λ' and α' are the representatives of $\dot{\lambda}$ and $\dot{\alpha}$ whose closed leaves all have a weight less than or equal to π). As we will see below the action of $Mod_0(M)$ on $\mathscr{P}^+(M)$ is properly discontinuous. Therefore $\mathscr{P}^+(M)/(Mod_0(M) * \mathscr{R})$ is a Hausdorff space. In this context, we have the following corollary of Theorem 1.2 :

Corollary. *Let $(\rho_n) \in GF(M)$ be a sequence of representations of $\pi_1(M)$, let $\mathring{\lambda}_n \in \mathscr{ML}(\partial M)/Mod_0(M)$ be the bending measured geodesic lamination of ρ_n and let $\ddot{\lambda}_n \in \mathscr{P}^+(M)/(Mod_0(M) * \mathscr{R})$ be the projection of $\mathring{\lambda}_n$. If the sequence $(\ddot{\lambda}_n)$ converges to some $\ddot{\lambda}_\infty \in \mathscr{P}^+(M)/(Mod_0(M) * \mathscr{R})$, then (ρ_n) contains a subsequence that converges strongly to a geometrically finite representation $\rho_\infty \in AH(M)$.*

It follows from the result of [Le3] that the bending measured geodesic lamination of ρ_∞ is (modulo the action of $Mod_0(M)$) the representative of $\ddot{\lambda}_\infty$ whose closed leaves all have a weight less than or equal to π.

Let $\mathring{\lambda} \in \mathscr{P}(M)/Mod_0(M)$ be a class of measured geodesic laminations and let $\mathscr{B}(\mathring{\lambda})$ be the set of geometrically finite representations $\rho \in AH(M)$ whose bending measured geodesic lamination is $\mathring{\lambda}$. Conjecturally $\mathscr{B}(\mathring{\lambda})$ is a single point, this conjecture has been proved in some special cases, see [BoO] and [Bo5]. Combining the corollary above and the results of [Ta] and [Le3] we get the following criterion for strong convergence.

Theorem 1.3. *Let $\mathring{\lambda} \in \mathscr{P}(M)/Mod_0(M)$ be a class of measured geodesic laminations such that $\mathscr{B}(\mathring{\lambda})$ is a single point ρ and let $\rho_n \subset AH(M)$ be a sequence of conjugacy classes of representations. Then ρ_n converges strongly to ρ_∞ if and only if :*

1. *there is N such that for $n \geq N$, ρ_n is a geometrically finite representation with bending measured geodesic lamination $\mathring{\lambda}_n \in \mathscr{P}(M)/Mod_0(M)$;*
2. *the sequence $\ddot{\lambda}_n \in \mathscr{P}^+(M)/(Mod_0(M) * \mathscr{R})$ converges to $\ddot{\lambda}$.*

Theorem 1.1 may also be used to study the action of the modular group on $\mathscr{ML}(\partial M)$. The modular group $Mod(M)$ is the group of isotopy classes of diffeomorphisms of M.

A measured geodesic lamination $\lambda \in \mathscr{ML}(\partial M)$ is *doubly incompressible* if $\exists \eta > 0$ such that $i(\partial E, \lambda) > \eta$ for any essential annulus or disc E. Let us denote by $\mathscr{D}(M)$ the set of doubly incompressible measured geodesic laminations. We have :

Theorem 1.4. *The action of $Mod(M)$ on $\mathscr{D}(M)$ is properly discontinuous.*

The paper is organized as follows. In section 2, we state some definitions and facts about geodesic laminations, convergence of representations, $\mathscr{P}(M)$ and $\mathscr{P}^+(M)$. In sections 3 and 4 we recall some ideas of [BoO] and [Le1] and complete them to prove the properness of $b_{\mathscr{R}}$. At this point, the proof of Theorems 1.1 and 1.2 is complete. In section 5, we deduce Theorem 1.4 from Theorem 1.1.

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2 Definitions

2.1 Geodesic laminations

Let $\partial_{\chi<0}M$ be the union of the components of ∂M with negative Euler characteristic. Assume that each component of $\partial_{\chi<0}M$ is endowed with a complete hyperbolic metric. A *geodesic lamination* on ∂M is a compact subset which is the disjoint union of complete simple geodesics embedded in $\partial_{\chi<0}M$. The space of geodesic laminations on ∂M endowed with the Hausdorff topology is denoted by $\mathscr{L}(\partial M)$. A geodesic lamination whose leaves are all closed is called a *multi-curve*. If each half-leaf of a geodesic lamination L is dense in L, then L is *minimal*. Such a minimal geodesic lamination is either a simple closed curve or an *irrational lamination*. A leaf l of a geodesic lamination L is *recurrent* if it lies in a minimal geodesic lamination. Any geodesic lamination is the disjoint union of finitely many minimal laminations and non-recurrent leaves. A leaf is said to be an *isolated* leaf if it is either a non-recurrent leaf or a compact leaf without any leaf spiraling toward it.

We say that two geodesic laminations L_1 and L_2 *intersect transversely* if a leaf of L_1 intersects a leaf of L_2 transversely. A geodesic lamination L is *arational* if any closed geodesic intersects L transversely.

It turns out that the definition of $\mathscr{L}(\partial M)$ can be made independent of the choice of the metric on $\partial_{\chi<0}M$. This is explained in the following note.

Note. Let us consider two complete hyperbolic metrics s_1 and s_2 on a connected component S of $\partial_{\chi<0}M$ and let $q : \mathbb{H}^2 \to S$ be a covering map. Let $l \subset S$ be a s_1-geodesic and let $\hat{l} \subset q^{-1}(l)$ be a lift of l. There is a unique s_2-geodesic $\Theta(l)$ such that there is a bounded homotopy between a lift $\hat{\Theta}(\hat{l}) \subset q^{-1}(\Theta(l))$ of $\Theta(l)$ and $\hat{l}$. Here a bounded homotopy is a map $F : \mathbb{R} \times [0,1] \to S$ such that the lengths of the arcs $F(\{x\} \times [0,1])$ are uniformly bounded. So we get a homeomorphism $\Theta : \{\text{geodesics of } (S, s_1)\} \longrightarrow \{\text{geodesics of } (S, s_2)\}$ and the image $\Theta(L)$ of a s_1-geodesic lamination L is a s_2-geodesic lamination. We will say that $\Theta(L)$ is the geodesic lamination L for the metric s_2. Thus the geodesic lamination L is well defined for any hyperbolic metric on $\partial_{\chi<0}M$.

Let L_1 and L_2 be two s_1-geodesic laminations and let $x \subset L_1 \cap L_2$ be a transverse intersection. Let $\hat{x} \subset \mathbb{H}^2$ be a lift of x. This point $\hat{x}$ is the intersection of a leaf $\hat{l}_1$ of $q^{-1}(L_1)$ and of a leaf $\hat{l}_2$ of $q^{-1}(L_2)$. Since $\hat{\Theta}(\hat{l}_1)$ intersects $\hat{\Theta}(\hat{l}_2)$ transversely, there is a map $L_1 \cap L_2 \to \Theta(L_1) \cap \Theta(L_2)$ which to x associates $q(\hat{\Theta}(\hat{l}_1) \cap \hat{\Theta}(\hat{l}_2))$. This way, each point lying in the transverse intersection of two geodesic laminations is defined independently of the choice of the metric on $\partial_{\chi<0}M$. $\diamond$

Let L be a connected geodesic lamination which is not a simple closed curve and let $\bar{S}(L)$ be the smallest surface with geodesic boundary containing L. This surface $\bar{S}(L)$ contains finitely many disjoint simple closed geodesics that are disjoint from L. We denote by $\partial'\bar{S}(L)$ the union of these geodesics (thus $\partial\bar{S}(L) \subset \partial'\bar{S}(L)$) . Removing annular neighborhoods of these curves from $\bar{S}(L)$ yields a surface which we call the *surface embraced* by L and which we denote by $S(L)$

(see [Le1] for details). When L is a closed curve we take an annular neighborhood for $S(L)$ and we define $\partial' \bar{S}(L)$ to be L. When L is not connected we take $S(L)$ to be the disjoint union of the surfaces embraced by the connected components of L.

A leaf l of a geodesic lamination L is homoclinic if any lift $\tilde{l}$ of l to the universal cover $\tilde{M}$ of M contains two sequences of points $\tilde{x}_n$ and $\tilde{y}_n$ such that the distance between $\tilde{x}_n$ and $\tilde{y}_n$ measured on $\tilde{l}$ tends to ∞ whereas their distance measured on M is uniformly bounded.

Let Λ be an ordered subgroup of $\mathbb{R}^p$ (with the lexicographic order) of rank p. A Λ*-measured lamination* $\mathscr{L}$ is a transverse measure with value in Λ for some geodesic lamination $|\mathscr{L}|$. Any arc $k \approx [0,1]$ embedded in ∂M transversely to $|\mathscr{L}|$, such that $\partial k \subset \partial M - |\mathscr{L}|$, is endowed with a finitely additive measure $d\mathscr{L}$ with values in Λ such that :

- the support of $d\mathscr{L}_{|k}$ is $|\mathscr{L}| \cap k$;
- if an arc k' can be homotoped into k by a homotopy respecting $|\mathscr{L}|$ then $\int_k d\mathscr{L} = \int_{k'} d\mathscr{L}$.

Let k be an embedded arc in S transverse to $|\mathscr{L}|$ satisfying $\partial k \subset S - |\mathscr{L}|$. We will denote by $[\int_k d\mathscr{L}]$ the first (counting from left to right) non-zero coordinate of $\int_k d\mathscr{L} \in \Lambda \subset \mathbb{R}^p$.

If λ is an $\mathbb{R}$-measured lamination (we will use Greek letters to name $\mathbb{R}$-measured laminations), then its support $|\lambda|$ does not contain any non-recurrent leaves. If γ is a weighted simple closed curve with weight $w(\gamma)$, the *intersection number* $i(\lambda,\gamma)$ is defined by $i(\lambda,\gamma) = w(\gamma)\int_{|\gamma|} d\lambda$. This extends to a continuous function $i : \mathscr{ML}(\partial M) \times \mathscr{ML}(\partial M) \to \mathbb{R}$ (cf. [Bo1]).

We will say that two Λ-measured laminations intersect transversely if their supports do so.

Let us now define the dual Λ-tree $\mathscr{T}_{\mathscr{L}}$ of a Λ-measured lamination $\mathscr{L} \subset S$ (cf. [MoO]). Let $q : \mathbb{H}^2 \to S$ be the covering projection. Let $pre\mathscr{T}_{\mathscr{L}}$ be the set whose points are the complementary regions of $q^{-1}(\mathscr{L})$ in $\mathbb{H}^2$. Let us define a distance $d : pre\mathscr{T}_{\mathscr{L}} \times pre\mathscr{T}_{\mathscr{L}} \to \Lambda$ in $pre\mathscr{T}_{\mathscr{L}}$. Let R_0 and R_1 be complementary regions and choose a geodesic segment $k \subset \mathbb{H}^2$ whose endpoints lie in R_0 and R_1 respectively. We define $d(R_0,R_1)$ to be the $q^{-1}(\mathscr{L})$-measure of k. There is a unique (up to isometry) Λ-tree $\mathscr{T}_{\mathscr{L}}$ and an isometric embedding $e : pre\mathscr{T}_{\mathscr{L}} \to \mathscr{T}_{\mathscr{L}}$ such that any point of $\mathscr{T}_{\mathscr{L}}$ lies in a segment with endpoints in $e(pre\mathscr{T}_{\mathscr{L}})$ (cf. [GiS]). This tree $\mathscr{T}_{\mathscr{L}}$ is the *dual tree* of $\mathscr{L}$. The covering transformations yield an isometric action of $\pi_1(M)$ on $\mathscr{T}_{\mathscr{L}}$.
Let c be a simple closed curve and let us denote by $\delta_{\mathscr{L}}(c)$ the distance of translation of an isometry of $\mathscr{T}_{\mathscr{L}}$ corresponding to c. We have then $\delta_{\mathscr{L}}(c) = \inf\{\int_{c'} d\mathscr{L} / c'$ is a closed curve freely homotopic to c and transverse to $|\mathscr{L}|\}$.

Consider now the case of an $\mathbb{R}$-measured lamination λ. This construction yields a natural projection $\mathbb{H}^2 - q^{-1}(\lambda) \to \mathscr{T}_\lambda$. If λ does not have any closed leaves, this projection extends continuously to a map $\pi_\lambda : \mathbb{H}^2 \to \mathscr{T}_\lambda$. Otherwise we replace the closed leaves of λ by foliated annuli endowed with uniform transverse measures, thus we get a continuous map $\pi_\lambda : \mathbb{H}^2 \to \mathscr{T}_\lambda$.

Let us come back to the general case where Λ is a subgroup of $\mathbb{R}^n$. The following claim links $\delta_{\mathscr{L}}(c)$ with the transverse measure of $\mathscr{L}$ for a simple closed curve $c \subset |\mathscr{L}|$.

Claim 2.1. *The measure $\delta_{\mathscr{T}_{\mathscr{L}}}(c)$ of a simple closed curve $c \subset |\mathscr{L}|$ is the half-sum of the measures supported by the half-leaves of $\mathscr{L}$ which spiral toward c.*

Proof. Let $\hat{c}$ be a lift of c in $\mathbb{H}^2$ and let g be the element of $\pi_1(S)$ that fixes $\hat{c}$. Denote by $q : \mathbb{H}^2 \to S$ the covering projection. If c is an isolated leaf, then g fixes the components of $\mathbb{H}^2 - q^{-1}(\mathscr{L})$ whose boundary contains $\hat{c}$. Therefore $\delta_{\mathscr{T}_{\mathscr{L}}}(c) = 0$. Otherwise let $\hat{x}$ be a point lying on a component of $\mathbb{H}^2 - q^{-1}(|\mathscr{L}|)$ whose closure contains an end ξ of $\hat{c}$. Denote by $[\hat{x}, g\hat{x}]$ the geodesic segment joining $\hat{x}$ to $g\hat{x}$ and let $\hat{d} = \bigcup_{i\in\mathbb{Z}} [g^i\hat{x}, g^{i+1}\hat{x}]$. This curve $\hat{d}$ is fixed by g. It is easy to see that the image of $\hat{d} - q^{-1}(|\mathscr{L}|)$ under the projection $\mathbb{H}^2 - q^{-1}(|\mathscr{L}|) \to \mathscr{T}_{\mathscr{L}}$ lies on the axis of g. Thus the distance of translation of g in $\mathscr{T}_{\mathscr{L}}$ is the measure of $[\hat{x}, g\hat{x}]$. The claim follows. □

A *train track* τ in ∂M is a finite union of rectangles which are foliated by vertical segments and which meet only at non-degenerate segments lying in the vertical sides. The rectangles are called *branches*, the horizontal segments are called *rails*, the vertical segments *ties* and the connected components of the union of the vertical sides are the *switches.* A *subtrack* τ' of a train track τ is a train track whose branches are all branches of τ. A geodesic lamination L is *carried by a train track* τ if L lies in τ and is transverse to the ties; a branch b of τ *carries* L if $L \cap b \neq \emptyset$. The train track τ *minimally carries* L if no proper subtrack of τ carries L.

We will conclude this subsection with some results about $\mathscr{ML}(\partial M)/\mathscr{R}$. The proofs of these results are quite easy and can be found in [Le3].

Let us denote by $\mathscr{ML}(\partial M)$ the space of $\mathbb{R}$-measured laminations (which we will simply call measured geodesic laminations) on $\partial_{\chi<0} M$ endowed with the weak* topology. Let $\mathscr{R}$ be the relation defined in the introduction. If $\lambda \mathscr{R} \mu$, then λ and μ have the same support. Therefore we can define the support $|\dot{\lambda}|$ of an element $\dot{\lambda}$ of $\mathscr{ML}(\partial M)/\mathscr{R}$ as the support of any representative of $\dot{\lambda}$.

When $\mathscr{ML}(\partial M)/\mathscr{R}$ is endowed with the quotient topology, we have the following :

Claim 2.2. *Let (λ_n) be a sequence of measured geodesic laminations such that $(\dot{\lambda}_n)$ tends to $\dot{\lambda}$ and that $|\lambda_n|$ tends to L in the Hausdorff topology. Then we have $|\dot{\lambda}| \subset L$.*

Claim 2.3. *Let $\lambda \in \mathscr{ML}(\partial M)$ and let λ' be the representative of $\dot{\lambda}$ whose compact leaves have all a weight less than or equal to π. Let $k \subset \partial M$ be a simple arc such that the points of $k \cap |\dot{\lambda}|$ are transverse intersections. Then we have $\int_k d\lambda' \leq \underline{\lim} \int_k d\lambda_n$. Furthermore, if k does not intersect a leaf of λ' with a weight equal to π, then $\int_k d\lambda_n$ converges to $\int_k d\lambda$.*

2.2 Algebraic and strong convergence

Let σ be a hyperbolic metric (up to isotopy) on $int(M)$. Given an isometry from the interior of $\tilde{M}$ to $\mathbb{H}^3$, the covering transformations yield a discrete faithful representation $\rho : \pi_1(M) \to Isom(\mathbb{H}^3)$. The representations which appear in this way will be called the *representations associated to* σ. The set of representations associated to σ is the set of all representations conjugate to ρ. The image $\rho(\pi_1(M))$ is a finitely generated torsion free Kleinian group and $int(M)$ endowed with σ is isometric to $\mathbb{H}^3/\rho(\pi_1(M))$. The *Nielsen core* of $\mathbb{H}^3/\rho(\pi_1(M))$ is the quotient by $\rho(\pi_1(M))$ of the convex hull $C(\rho)$ of the limit set L_ρ of $\rho(\pi_1(M))$ (see [Th1, chap 8] for details). This set $N(\rho)$ and the convex core defined in the introduction are isometric and from now on we will identify them. The *thick part*, $N(\rho)^{ep}$ of the Nielsen core is the complement of the cuspidal part of $\mathbb{H}^3/\rho(\pi_1(M))$ in $N(\rho)$. The representation ρ is *geometrically finite* when $N(\rho)^{ep}$ is compact (here it is equivalent to say that $N(\rho)$ has finite volume) and *convex cocompact* when $N(\rho)$ is compact.

The boundary of $N(\rho)^{ep}$ is the union of tori and compact surfaces. Any such surface is the union of annuli lying in the boundary of the cuspidal part and of surfaces lying in the boundary of $N(\rho)$. These last surfaces are bent along a geodesic lamination (cf. [Th1, chap 8]). The *bending locus* of $\rho(\pi_1(M))$ is the union of these geodesic laminations $|\lambda^i|$ and of the cores of the annuli corresponding to the rank 1 cusps. This bending locus is endowed with a transverse measure measuring the amount of bending along the geodesic laminations $|\lambda^i|$ and endowing the cores of the annuli with Dirac measures of weight π (cf. [EpM]). When ρ is geometrically finite and not fuchsian, the natural retraction from $\mathbb{H}^3/\rho(\pi_1(M))$ to $N(\rho)$ associates to σ a homeomorphism (defined up to isotopy) $h : M \to N(\rho)^{ep}$. Any homeomorphism $M \to N(\rho)^{ep}$ isotopic to h is said to be *associated* to σ. By this homeomorphism h, we may interpret the bending locus of $\partial N(\rho)$

together with its transverse measure as a measured geodesic lamination $b(\sigma) \in \mathscr{ML}(\partial M)$, depending only on the isotopy class of σ. This is the bending measured geodesic lamination of the geometrically finite metric σ.

Let (σ_n) be a sequence of complete hyperbolic metrics on the interior of M. This sequence (σ_n) *converges algebraically* when we can choose a sequence of representations $\rho_n : \pi_1(M) \to Isom(\mathbb{H}^3)$ associated to the σ_n (as above) that converges algebraically, namely $\rho_n(g)$ converges for any $g \in \pi_1(M)$. Defining $\rho_\infty(g)$ by $\rho_\infty(g) = \lim_{n\longrightarrow\infty} \rho_n(g)$, we obtain a new representation $\rho_\infty : \pi_1(M) \to Isom(\mathbb{H}^3)$ which is discrete and faithful (cf. [Jor]). This representation defines a metric on a manifold homotopy equivalent to M. This manifold might not be homeomorphic to M (some examples are given in [AnC]).

The sequence (σ_n) *converges geometrically* if we can choose a sequence of representations $\rho_n : \pi_1(M) \to Isom(\mathbb{H}^3)$ associated to the σ_n so that $(\rho_n(\pi_1(M))$ converges geometrically. Namely there is $\Gamma_\infty \subset Isom(\mathbb{H}^3)$ such that

- for any sequence $a_n \in \rho_n(\pi_1(M))$, any accumulation point a of $\{a_n/n \in \mathbb{N}\}$ lies in Γ_∞
- any element a of Γ_∞ is the limit of a sequence $a_n \in \Gamma_n$.

The limit manifold $\mathbb{H}^3/\Gamma_\infty$ can be different from $int(M)$.

The sequence $\rho_n(\pi_1(M))$ *converges strongly* if we can choose the representations ρ_n so that ρ_n converges algebraically to some representation ρ_∞ and so that $(\rho_n(\pi_1(M)))$ converges geometrically to $\rho_\infty(\pi_1(M))$. If (σ_n) converges to σ in the topology defined in the introduction, then (σ_n) converges strongly to σ_∞ (cf. [CEG]).

2.3 Some relations between $\mathscr{P}(M)$, $\mathscr{P}^+(M)$ and $\mathscr{D}(M)$

Let $\mathscr{P}(M)$ denote the set of measured geodesic laminations satisfying $a)$, $b)$ and $c)$ and let $\mathscr{P}^+(M)$ denote the set of measured geodesic laminations satisfying $b)$, $c)$ and $d)$. By definition $\mathscr{P}^+(M)$ is saturated in $\mathscr{R}$. It is easy to see that $a)$ and $c)$ together implies $d)$ and hence that $\mathscr{P}(M) \subset \mathscr{P}^+(M)$. The injectivity of the map $\mathscr{P}(M) \to \mathscr{P}^+(M)/\mathscr{R}$ is a direct consequence of condition $a)$. Its surjectivity comes from the following lemma.

Lemma 2.4. *Let $\lambda \subset \mathscr{P}^+(M)$ be a measured geodesic lamination. Then the measured geodesic lamination λ' obtained by replacing by π the weights of leaves of λ which have a weight greater than π belongs to $\mathscr{P}(M)$.*

Proof. This lemma is a straightforward consequence of the proof of [Le2, Lemma 3.4]. □

A measured geodesic lamination $\lambda \in \mathscr{ML}(\partial M)$ is *doubly incompressible* if there is $\eta > 0$ such that $i(\lambda, \partial F) > \eta$ for any essential annulus or disk F. We will denote by $\mathscr{D}(M)$ the set of doubly incompressible measured geodesic laminations. We have the following :

Lemma 2.5. *Let λ be a measured geodesic lamination lying in $\mathscr{P}(M)$. For each irrational sublamination λ_i of λ, let us remove from λ its closed leaves lying in $\partial' \bar{S}(\lambda)$. Let γ be the geodesic measured lamination thus obtained. Then γ lies in $\mathscr{D}(M)$.*

Proof. First assume that, up to extracting a subsequence, there is a sequence of essential discs D_n such that $i(\gamma, \partial D_n) \longrightarrow 0$. Extract a subsequence such that ∂D_n converge for the Hausdorff topology to a geodesic lamination L. We have $i(\gamma, \partial D_n) \longrightarrow 0$, so L does not intersect γ transversely. By Casson's criterion (cf. [Ot3], [Le1, Theorem B1]), L contains a homoclinic leaf l. By [Le1, Lemma 3.4], l intersects λ transversely. Since l does not intersect γ transversely, l intersects $\lambda - \gamma$ once or twice. Let l_+ and l_- be the two unbounded connected components of $l - \lambda$. Since l is a homoclinic leaf, there is a bounded homotopy in M between l_+ and l_-. Since l_+ and l_- do not intersect λ transversely, it follows from [Le1, Proposition 3] that l_+ and l_- are asymptotic on ∂M. Therefore there is an arc $k \in \lambda - l$ such that $k \cup l_+ \cup l_-$ bounds an ideal triangle in ∂M. Set $\kappa = l - (l_+ \cup l_-)$, then $\kappa \cup k$ bounds an essential disc and $i(\lambda, \kappa \cup k) = 0$. This contradicts the fact that λ satisfies condition c). Thus we have proved that there is $\eta > 0$ such that for any essential disc D, $i(\lambda, \partial D) \leq \eta$.

Assume now that there is a sequence of essential annuli A_n such that $i(\gamma, \partial A_n)$ tends to 0. Let us extract a subsequence such that ∂A_n converge in the Hausdorff topology to a geodesic lamination L. This lamination L does not intersect γ transversely. Let $p : \tilde{M} \to M$ be the covering projection. It follows from the proof of [Le1, Lemma C2] that $p^{-1}(L)$ contains two leaves $\tilde{l}_1$ and $\tilde{l}_2$ such that there is a bounded homotopy in M between $\tilde{l}_1$ and $\tilde{l}_2$. Since L does not intersect γ transversely, $\tilde{l}_1$ and $\tilde{l}_2$ intersect $p^{-1}(\lambda)$ finitely many times (actually at most twice). For $i = 1, 2$, let $\tilde{l}_i^+$ and $\tilde{l}_i^-$ be the unbounded components of $\tilde{l}_i - p^{-1}(\lambda)$. If $\tilde{\kappa}_i = \tilde{l}_i - (\tilde{l}_i^+ \cup \tilde{l}_i^-)$ is not a point, its interior does not intersect λ. From the bounded homotopies between $\tilde{l}_1$ and $\tilde{l}_2$ we deduce bounded homotopies in M between $\tilde{l}_1^+$ and $\tilde{l}_2^+$ and between $\tilde{l}_1^-$ and $\tilde{l}_2^-$ respectively. It follows then from [Le1, Proposition 3] that $\tilde{l}_1^+$ and $\tilde{l}_2^+$ are asymptotic on $\partial\tilde{M}$ and that $\tilde{l}_1^-$ and $\tilde{l}_2^-$ are asymptotic in $\partial\tilde{M}$. This implies that there are two arcs $\tilde{k}^+, \tilde{k}^- \in p^{-1}(\lambda) - p^{-1}(L)$ such that $\tilde{k}^+ \cup \tilde{l}_1^+ \cup \tilde{l}_2^+$ and $\tilde{k}^- \cup \tilde{l}_1^- \cup \tilde{l}_2^-$ bound two ideal triangles in $\partial\tilde{M}$. Since $\tilde{l}_1$ and $\tilde{l}_2$ are not homotopic in $\partial\tilde{M}$, neither $\tilde{\kappa}_1$ nor $\tilde{\kappa}_2$ is a point. The curve $\tilde{c} = \tilde{k}^+ \cup \tilde{\kappa}_1 \cup \tilde{k}^- \cup \tilde{\kappa}_2$ bounds an essential disc and we have $i(c, \lambda) = 0$. This contradicts the assumption that $\lambda \in \mathscr{P}(M)$ and concludes the proof. □

3 Convergence of bending laminations and algebraic convergence of metrics

We will begin by proving that the map $b_{\mathscr{R}}$ is proper in the algebraic topology.

Proposition 3.1. *Let M be a compact, orientable, hyperbolic 3-manifold with boundary and let (σ_n) be a sequence of geometrically finite metrics on the interior of M. Let $\lambda_n \subset \mathscr{ML}(\partial M)$ be the bending measured geodesic lamination of σ_n. Suppose that $\dot{\lambda}_n \in \mathscr{ML}(\partial M)/\mathscr{R}$ tends to $\dot{\lambda} \subset \mathscr{P}^+(M)/\mathscr{R}$. Then a subsequence of (σ_n) converges algebraically.*

Proof. Let us assume the contrary, namely that there is no sequence ρ_n of representations associated to the σ_n which contains an algebraically converging subsequence. By the theory of [MoS1], there exists a sequence (ε_n) of positive numbers tending to 0, a small minimal action of $\pi_1(M)$ on an $\mathbb{R}$-tree $\mathscr{T}$ and a subsequence of (ρ_n) (which will also be denoted by (ρ_n)) such that, for any $g \in \pi_1(M)$, we have $l_{\mathscr{T}}(g) = \lim_{n \longrightarrow \infty} \varepsilon_n l_{\rho_n}(g)$. Here $l_{\rho_n}(g)$ is the distance of translation of $\rho_n(g)$ and $l_{\mathscr{T}}(g)$ is the distance of translation of g acting on $\mathscr{T}$.

In order to prove Proposition 3.1, we will need some results about realizations of laminations in $\mathbb{R}$-trees.

3.1 Realizations of laminations in $\mathbb{R}$-trees

Let $\pi_1(M) \times \mathscr{T} \to \mathscr{T}$ be a small minimal action on an $\mathbb{R}$-tree $\mathscr{T}$, let $L \subset \partial M$ be a geodesic lamination and let S be the connected component of ∂M containing L. The lamination L is *realized in* $\mathscr{T}$ if there is a continuous $\pi_1(M)$ equivariant map $\hat{S} \to \mathscr{T}$ which is injective when restricted to any lift of a leaf of L to the universal cover $\hat{S}$ of S.

The following result comes from the proof of [Le1, Proposition 6] (cf. [Le2]).

Proposition 3.2 ([Le2], Proposition 6.1). *Let $\mathscr{T}$ be an $\mathbb{R}$-tree, let $\pi_1(M) \times \mathscr{T} \to \mathscr{T}$ be a small minimal action and let $\lambda \in \mathscr{D}(M)$. Then at least one connected component of λ is realized in $\mathscr{T}$.*

Let λ be a measured geodesic lamination. A geodesic lamination L is called an *expansion* of λ if $|\lambda| \subset L$ and if $L - |\lambda|$ is a union of finitely many non-recurrent leaves, each one going from an irrational component of $|\lambda|$ to itself.

Using Lemma 2.5 and Proposition 3.2, we get the following :

Proposition 3.3. *Let $\lambda \subset \mathscr{P}^+(M)$ and let E be an expansion of $|\lambda|$. Let $\pi_1(M) \times \mathscr{T} \to \mathscr{T}$ be a small minimal action on an $\mathbb{R}$-tree $\mathscr{T}$. Then there is at least one connected component E_1 of E which is realized in $\mathscr{T}$ and which satisfies one of the following conditions:*

- *1) E_1 contains an irrational geodesic lamination;*
- *2) E_1 is a closed leaf and E_1 does not lie in $\partial'\bar{S}(E_i)$ for any component $E_i \neq E_1$ of E.*

Proof. Let E_{rec} be the union of the recurrent leaves of E. Let α be a measured geodesic lamination whose support is E_{rec}. Let us remove from α the closed leaves lying in $\partial'\bar{S}(\alpha_i)$ for some irrational sublamination α_i of α and let γ be the resulting measured geodesic lamination. By Lemma 2.5, γ lies in $\mathscr{D}(M)$. By Proposition 3.2, at least one connected component γ_1 of γ is realized in $\mathscr{T}$. Let E_1 be the connected component of E containing the support of γ_1. By [Ot2, chap. 3], E_1 is realized in $\mathscr{T}$. By construction E_1 satisfies 1) or 2). □

In the proof of Proposition 3.1, we are interested in actions of $\pi_1(M)$ on $\mathscr{T}$ coming from a degenerating sequence of representations $\rho_n(\pi_1(M))$. In [Ot2], J.-P. Otal described, in the special case of handlebodies, the behavior of the lengths of measured geodesic laminations which are realized in $\mathscr{T}$. A careful look at the proof yields the following statement.

Theorem 3.4 (Continuity Theorem [Ot2]). *Let (ρ_n) be a sequence of geometrically finite representations of $\pi_1(M)$ tending to a small minimal action of $\pi_1(M)$ on an $\mathbb{R}$-tree $\mathscr{T}$. Let $\varepsilon_n \longrightarrow 0$ be such that $\forall g \in \pi_1(M)$, we have $\varepsilon_n l_{\rho_n}(g) \longrightarrow l_{\mathscr{T}}(g)$ and let L be a geodesic lamination which is realized in $\mathscr{T}$. Then there exists a neighborhood $\mathscr{V}(L)$ and constants K, n_0 such that for any simple closed curve $c \subset \mathscr{V}(L)$ and for any $n \geq n_0$,*

$$\varepsilon_n l_{\sigma_n}(c^*) \geq K l_{s_0}(c).$$

When $c \subset \partial_{\chi<0} M$ is a closed curve, $l_{\sigma_n}(c^*)$ denote the length of the closed geodesic of $\mathbb{H}^3/\rho_n(\pi_1(M))$ in the free homotopy class of c.

3.2 Proof of Proposition 3.1

We have assumed that (ρ_n) does not contain an algebraically converging subsequence and we have deduced from [MoS1] that a subsequence of ρ_n tends to an action of $\pi_1(M)$ on an $\mathbb{R}$-tree $\mathscr{T}$. Let s_n be the metric on $\partial_{\chi<0} M$ induced by the composition of the inclusion map and of a homeomorphism $h_n : M \to N(\rho_n)$ associated to σ_n. The following lemma which will be proved in section 3.4 is an important step in our proof of Proposition 3.1.

Lemma 3.5. *Let (λ_n) be a sequence of geodesic measured laminations such that $(\dot{\lambda}_n)$ converges in $\mathscr{ML}(\partial M)/\mathscr{R}$ to $\dot{\lambda}$. There is an expansion $E \in \mathscr{L}(\partial M)$ of $|\lambda|$ such that for any connected component E_i of E one of the two following situations occurs :*

- *i) there is a sequence of weighted multi-curves γ_n^i such that $(|\gamma_n^i|)$ converges to E_i in the Hausdorff topology, that (γ_n^i) converges in $\mathscr{ML}(\partial M)$ to a non trivial lamination γ_∞^i and that we have $l_{s_n}(\gamma_n^i) \leq l_{s_n}(\lambda_n) + i(\lambda_n, \partial'\bar{S}(E_i)) l_{s_n}(\partial'\bar{S}(E_i))$;*
- *ii) E_i is a closed leaf of $|\dot{\lambda}|$.*

Let E_i be a component of E satisfying $i)$. We are going to prove that E_i is not realized in $\mathscr{T}$.

By [MoS1], for large n, we have $\varepsilon_n l_{\sigma_n}(\partial' \bar{S}(E_i)^*) \leq l_{\mathscr{T}}(\partial' \bar{S}(E_i)) + 1$.
Since $|\dot{\lambda}| \subset E$, Claim 2.3 implies that $i(\lambda_n, \partial' \bar{S}(E_i))$ tends to 0. By the slight bending Lemma ([Le1, Lemma A.1] see also [Se]) this allows us to approximate $l_{s_n}(\partial' \bar{S}(E_i))$ by $l_{\sigma_n}(\partial' \bar{S}(E_i)^*)$, namely $l_{s_n}(\partial' \bar{S}(E_i)) \leq A_{\eta_n}(l_{\sigma_n}(\partial' \bar{S}(E_i)^*) + B_{\eta_n})$ for some $A_{\eta_n} \longrightarrow 1$ and $B_{\eta_n} \longrightarrow 0$. Thus we obtain $\varepsilon_n l_{s_n}(\partial' \bar{S}(E_i)) \leq l_{\mathscr{T}}(\partial' \bar{S}(E_i)) + 1$ for large enough n. Replacing in the inequality of $i)$, we get

$$\varepsilon_n l_{s_n}(\gamma_n^i) \leq \varepsilon_n l_{s_n}(\lambda_n) + i(\lambda_n, \partial' \bar{S}(E_i))(l_{\mathscr{T}}(\partial' \bar{S}(E_i) + 1).$$

As shown in [BoO] and [Le1], $l_{s_n}(\lambda_n)$ is bounded :

Lemma 3.6. *Under the hypothesis of Proposition 3.1 the lengths $l_{\sigma_n}(\lambda_n)$ are bounded from above.*

Proof. The proof of [Le1, Lemma 4.1] extends to this situation.

□

Thus we have $\varepsilon_n l_{s_n}(\gamma_n^i) \leq \varepsilon_n l_{s_n}(\lambda_n) + i(\lambda_n, \partial' \bar{S}(E_i))(l_{\mathscr{T}}(\partial' \bar{S}(E_i) + 1) \longrightarrow 0$ (the fact $i(\lambda_n, \partial' \bar{S}(E_i)) \longrightarrow 0$ comes from Claim 2.3).

Since γ_∞^i is not trivial, we have $l_{s_0}(\gamma_n^i) \longrightarrow l_{s_0}(\gamma_\infty^i) > 0$. It follows that $\varepsilon_n \frac{l_{s_n}(\gamma_n^i)}{l_{s_0}(\gamma_n^i)}$ tends to 0. We deduce from the Otal's continuity theorem above that E_i is not realized in $\mathscr{T}$.

By Proposition 3.2, at least one component E_i of E is realized in $\mathscr{T}$ and E_i satisfies condition 1) or 2) of Proposition 3.2. By the above, E_i is a closed leaf of $|\dot{\lambda}|$. So E_i satisfies 2), namely E_i is not homotopic to a curve lying in the surface embraced by an irrational component of $\dot{\lambda}$. Since E_i is realized in $\mathscr{T}$, we have $\varepsilon_n l_{\sigma_n}(E_i^*) \longrightarrow l_{\mathscr{T}}(E_i) > 0$. The fact that E_i satisfies 2) allows us to find a surface $F \subset \partial M$ with geodesic boundary such that $E \cap int(F) = E_i$. Let $c \subset F$ be a simple closed geodesic intersecting E_i in one or two points. This geodesic c satisfies the following:

Lemma 3.7. *The simple closed geodesic c has the two following properties :*

- $\frac{l_{s_n}(E_i)}{l_{s_n}(c)} \longrightarrow 0$*;*
- $\exists K$ *such that* $l_{s_n}(c) \leq K l_{\sigma_n}(c^*)$.

Before proving Lemma 3.7 let us conclude the proof of Proposition 3.1. By Lemma 3.7, we have $\frac{l_{\sigma_n}(E_i^*)}{l_{\sigma_n}(c^*)} \leq K \frac{l_{s_n}(E_i)}{l_{s_n}(c)} \longrightarrow 0$. By [MoS1], $\varepsilon_n l_{\sigma_n}(c^*)$ is bounded. Therefore we have $\varepsilon_n l_{\sigma_n}(E_i^*) \longrightarrow 0$. This contradicts the fact that E_i is realized in $\mathscr{T}$ and concludes the proof of Proposition 3.1. □

3.3 Proof of Lemma 3.7

To prove the first property we are going to study the behavior of the sequence (s_n). Since $(\dot{\lambda}_n)$ converges, up to extracting a subsequence, we may assume that the leaves of λ_n which have a weight equal to π does not depend on n. Let us denote by $\lambda^{(p)}$ their union. Let S be the connected component of $\partial M - \lambda^{(p)}$ containing E_i and let us now denote by s_n the complete hyperbolic metric induced on S by a homeomorphism $h_n : M \to N(\rho_n)$ associated to σ_n. Since E_i is realized in $\mathscr{T}$, its length tends to infinity. So (s_n) is not bounded in Teichmüller space. By [MoS1] a subsequence of (s_n) tends to an action of $\pi_1(S)$ on a Λ-tree $\mathscr{T}_{\mathscr{L}}$. The basic property of $\mathscr{T}_{\mathscr{L}}$ is that for any two simple closed geodesics c_1, c_2 such that $l_{s_n}(c_1) \longrightarrow \infty$, we have

$\lim_{n\longrightarrow\infty}\frac{l_{s_n}(c_2)}{l_{s_n}(c_1)}=\frac{\delta_{\mathscr{T}_{\mathscr{L}}}(c_2)}{\delta_{\mathscr{T}_{\mathscr{L}}}(c_1)}$. By [MoO], this action is dual to a Λ-measured lamination $\mathscr{L}$. In this context, we have the following version of the continuity theorem (see the proof in appendix).

Theorem A.1. *Let (s_n) be a sequence of complete hyperbolic metrics of finite area on a surface S tending to a Λ-measured lamination $\mathscr{L}$ and let (λ_n) be a sequence of measured geodesic laminations such that (λ_n) converges to some $\dot{\lambda}\in\mathscr{ML}(S)/\mathscr{R}$ and such that $l_{s_n}(\lambda_n)$ is bounded. Then $|\dot{\lambda}|$ does not intersect $|\mathscr{L}|$ transversely.*

Since $l_{s_n}(\lambda_n)$ is bounded (Lemma 3.6), it follows that $|\dot{\lambda}|$ does not intersect $|\mathscr{L}|$ transversely. Since E_i is realized in $\mathscr{T}$, we have $l_{s_n}(E_i)\longrightarrow\infty$. By the construction of the Λ-tree $\mathscr{T}$, this implies that $\delta_{\mathscr{L}}(E_i)>0$. Therefore, by Claim 2.1, $|\mathscr{L}|$ contains E_i and at least one leaf spiraling toward E_i. It follows that $0<\delta_{\mathscr{L}}(E_i)<\delta_{\mathscr{L}}(c)$, hence we have $\frac{l_{s_n}(E_i)}{l_{s_n}(c)}\longrightarrow 0$.

The second property will come from the fact that c can be cut into arcs with bounded bending measure and unbounded length.

Claim 3.8. *For any sequence of arcs $\kappa_n\subset c$ with $\underline{\lim}\int_{\kappa_n}d\lambda_n>0$, we have $l_{s_n}(\kappa_n)\longrightarrow\infty$.*

Proof. We are going to compare the behavior of the sequences (s_n) and (λ_n) near λ^i. Let λ' be the representative of $\dot{\lambda}$ lying in $\mathscr{P}(M)$. If E_i is a leaf of λ_n, then its weight in λ_n converges to its weight in λ'. Since $l_{s_n}(E_i)$ tends to ∞ and $l_{s_n}(\lambda_n)$ is bounded, we have $E_i\not\subset|\lambda_n|$ for large n.

Let us consider a subsequence such that $(|\lambda_n|)$ converges to some geodesic lamination L' in the Hausdorff topology. Let $\mathscr{V}(E_i)$ be an annular neighborhood of E_i. By Claim 2.2, we have $E_i\subset L'$. Therefore, for large n, $\mathscr{V}(E_i)\cap|\lambda_n|$ is the union of disjoint segments joining the two components of $\partial\mathscr{V}(E_i)$ and turning a lot of times toward E_i (see figure 1). By Claim 2.3, we have

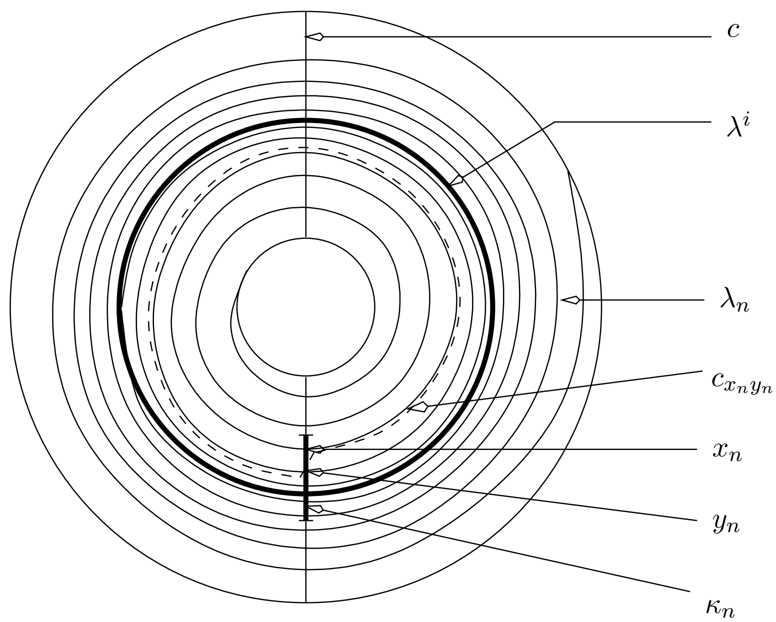

Figure 1: Picture of $\mathscr{V}(\lambda^i)$

$i(\lambda_n,E_i)\longrightarrow 0$. Therefore the total weight of the transverse measure carried by $|\lambda_n|\cap\mathscr{V}(E_i)$ tends to 0.

Let $\kappa_n\subset c$ be a sequence of segments such that $\underline{\lim}\int_{\kappa_n}d\lambda_n>0$. Since $c\cap|\dot{\lambda}|\subset E_i$, it follows from Claim 2.3 that $\int_{\kappa_n-\mathscr{V}(E_i)}d\lambda_n$ tends to 0. Therefore it is sufficient to prove Claim 3.8 for $\kappa_n\subset\mathscr{V}(E_i)$. We may also assume that $\partial\kappa_n\subset|\lambda_n|$. So $\partial\kappa_n$ lies in $|\lambda_n|\cap c$ and κ_n is well defined for any hyperbolic metric on $\partial_{\chi<0}M$ (see the note in section 1.1). Given an orientation of c we order the points of $\kappa_n \cap \lambda_n$ by the induced order. By the previous description of $\lambda_n \cap \mathscr{V}(E_i)$, for large n, κ_n contains an arc κ'_n such that $\int_{\kappa'_n} d\lambda_n \geq \frac{\int_{\kappa_n} d\lambda_n}{3}$ and such that any point x_n of $\kappa'_n \cap \lambda_n$ is joined to a point $y_n > x_n$ of $\kappa_n \cap \lambda_n$ by an arc $]x_n, y_n[\subset \lambda_n - \kappa_n$. Fix n and consider a point x_n in $\kappa'_n \cap \lambda_n$ and the associated point y_n. Let $c_{x_n y_n}$ denote the simple closed curve which is the union of the arc $[x_n, y_n] \subset \lambda_n$ and of the arc $\kappa_{x_n y_n} \subset \kappa_n$ joining x_n to y_n (see figure 1). This curve $c_{x_n y_n}$ is homotopic to E_i, hence $l_{s_n}(c_{x_n y_n})$ is greater than $l_{s_n}(E_i)$. It follows that $\forall n$, $\forall x_n \subset \kappa'_n \cap \lambda_n$, we have $l_{s_n}([x_n, y_n]) \geq l_{s_n}(E_i) - l_{s_n}(\kappa_n)$. Since $l_{s_n}(\lambda_n) \geq (\int_{\kappa'_n} d\lambda_n) \inf\{l_{s_n}([x_n, y_n]) \,|\, x_n \subset \kappa'_n \cap \lambda_n\}$, we have $l_{s_n}(\lambda_n) \geq (\int_{\kappa'_n} d\lambda_n)(l_{s_n}(E_i) - l_{s_n}(\kappa_n))$. Thus, we have $l_{s_n}(\kappa_n) \geq l_{s_n}(E_i) - \frac{3}{\int_{\kappa_n} d\lambda_n} l_{s_n}(\lambda_n)$. Since $l_{s_n}(E_i) \longrightarrow \infty$ and $l_{s_n}(\lambda_n)$ is bounded, we get $l_{s_n}(\kappa_n) \longrightarrow \infty$. □

Let $0 < \epsilon < \frac{i(c,\lambda')}{2}$. It follows from the description of $\lambda_n \cap \mathscr{V}(E_i)$ that, for large n, we may divide c into $\lfloor \frac{i(c,\lambda_n)}{\epsilon} \rfloor$ segments κ_n^j such that $\forall j \leq \lfloor \frac{i(c,\lambda_n)}{\epsilon} \rfloor$, we have $\epsilon \leq \int_{\kappa_n^j} d\lambda_n < 2\epsilon$. By Claim 3.8, $\forall j$, $l_{s_n}(\kappa_n^j) \longrightarrow \infty$. Given ϵ such that $2\epsilon \leq \frac{\pi}{3}$, let c_n be the union of the geodesic segments of (M, σ_n) joining the vertices of the segments κ_n^j. A consequence of the slight bending Lemma of [Le1] ([Le1, Lemma A.1]) is that $\exists K_\epsilon > 0$ such that, $l_{s_n}(c) \leq K_\epsilon l_{\sigma_n}(c_n)$. Moreover, by the Gauss-Bonnet formula, the geodesic arcs composing c_n have interior angles greater than $\pi - 2\frac{\pi}{3} = \frac{\pi}{3}$. Hence the curve c_n is the union of segments which are longer and longer (with respect to the metrics σ_n) and have interior angles greater than $\frac{\pi}{3}$. In this situation, it is a classical result (see [Ot3] for example) that $\exists K'_\epsilon$ such that we have $l_{\sigma_n}(c_n) \geq K'_\epsilon l_{\sigma_n}(c^*)$. Thus taking $K = K'_\epsilon K_\epsilon$ yields the second property of Lemma 3.7. This concludes the proof of Lemma 3.7. □

3.4 Construction of L

Let S be a connected component of $\partial_{\chi<0} M$. Fix n and consider a sequence (ν_k) of weighted simple closed curves converging in $\mathscr{ML}(S)$ to $\lambda_n \cap S$. Thus we have $l_{s_n}(\nu_k) \longrightarrow l_{s_n}(\lambda_n \cap S)$. Since $l_{s_n}(\lambda_n)$ is bounded (Lemma 3.6), using a diagonal extraction, one constructs a sequence of weighted simple closed geodesics (ν_n) such that $l_{s_n}(\nu_n)$ is bounded and that $(\dot{\nu}_n)$ tends to $\dot{\lambda} \cap S$. To simplify the formulae, we will assume that we have $l_{s_n}(\nu_n) \leq l_{s_n}(\lambda_n)$ and $i(\nu_n, \partial' \bar{S}(\lambda^i)) \leq i(\lambda_n, \partial' \bar{S}(\lambda^i))$ for any connected component λ^i of λ (this can be achieved by multiplying ν_n by a convenient constant $C_n < 1$ with $C_n \longrightarrow 1$).

Let λ^i be an irrational component of $\dot{\lambda}$ and let $S(\lambda^i)$ be the surface it embraces. We will change $\nu_n \cap S(\lambda^i)$ as shown in figure 2. This will yield the sequence of weighted multi-curves (γ_n^i) that appears in Lemma 3.5. This construction easily provides the control on $l_{s_n}(\gamma_n^i)$ that is needed in Lemma 3.5 but we have to take care to ensure that γ_n^i does not converge to the trivial measured geodesic lamination.

Let us extract a subsequence such that $|\dot{\nu}_n|$ converges to some geodesic lamination N. By Claim 2.2, $|\dot{\lambda}| \subset N$. It follows that $k = N \cap S(\lambda^i)$ is the union of λ^i, of simple geodesics asymptotic to λ^i and of simple half-geodesic which have an endpoint in $\partial' \bar{S}(\lambda^i)$ and which are asymptotic to λ^i. Let $c \subset S(\lambda^i)$ be a simple closed curve. Any leaf of N intersecting $S(\lambda^i)$ intersects c. Therefore there is a train track (with sidings) $\tau \subset \bar{S}(\lambda^i)$ such that :

- τ minimally carries $N \cap S(\lambda^i)$, namely $N \cap S(\lambda^i) \subset \tau$ and each branch of τ carries N;
- τ has one switch $A \subset c$ and the remaining ones lie in $\partial' \bar{S}(\lambda^i)$ (they correspond to sidings);
- any two arcs $k^1, k^2 \subset k - A$ such that $\bar{k}^1$ and $\bar{k}^2$ are isotopic relatively to $A \cup \partial' \bar{S}(\lambda^i)$ lie in the same rectangle of τ.

Such a train track may be constructed in the following way. Let $\tau' \subset \bar{S}(\lambda^i)$ be a train track (with sidings) carrying $N \cap S(\lambda^i)$ such that one switch A of τ' lies in c and that the remaining

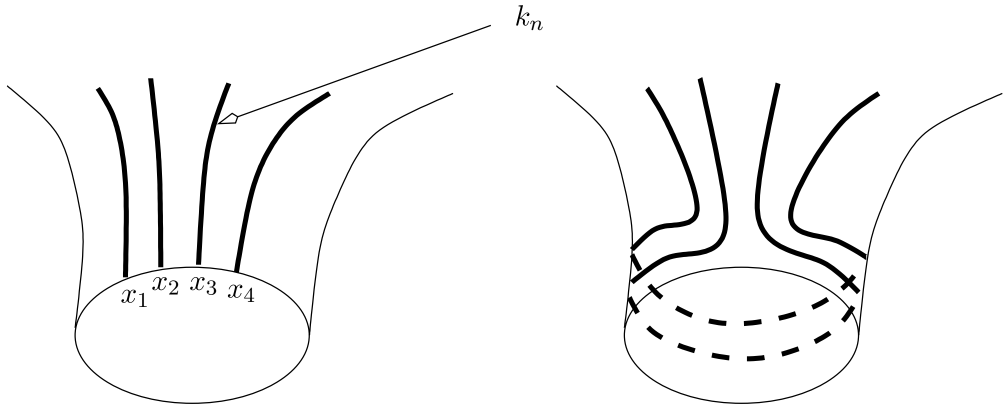

Figure 2: Constructing a multi-curve from k_n

switches lie in $\partial'\bar{S}(\lambda^i)$ (see [Bo1] or [Ot3] for the construction of τ'). Let b_1 and b_2 be two branches of τ' containing two arcs $k_1 \subset k \cap b_1$ and $k_2 \subset k \cap b_2$ which are isotopic relatively to $A \cup \partial'\bar{S}(\lambda^i)$. There are two arcs $\kappa_1, \kappa_2 \subset A \cup \partial'\bar{S}(\lambda^i)$ such that $k_1 \cup \kappa_1 \cup k_2 \cup \kappa_2$ bounds a disc $D \subset \bar{S}(\lambda^i)$. Let k_3 be a connected component of $k \cap D$, then ∂k_3 lies in $\kappa_1 \cup \kappa_2$. Since κ_1, κ_2 and k_3 are geodesic arcs, the two vertices of k_3 do not lie in the same segment κ_i. Hence k_3 is isotopic to k_1 relatively to $\kappa_1 \cup \kappa_2 \subset A \cup \partial'\bar{S}(\lambda^i)$. Replace k_2 by k_3 and b_2 by the branch of τ' that contains k_3 and repeat this operation until $D - (b_1 \cup b_2) \cap k = \emptyset$. Then extend the foliation of b_1 and b_2 to a foliation of $b_1 \cup D \cup b_2$ and replace b_1 and b_2 by the new branch $b_1 \cup D \cup b_2$ thus constructed. Repeating this process finitely many times yields the desired train track τ.

Fix n large enough so that $|\nu_n| \cap \bar{S}(\lambda^i)$ is carried by τ and denote by k_n the union of two parallel copies of $|\nu_n| \cap \bar{S}(\lambda^i)$ carried by τ. Let d be a connected component of $\partial'\bar{S}(\lambda^i)$. The integer $\sharp(d \cap k_n)$ is even. Choose x_1 such that when $x_1, x_2, ..., x_{2p}$ are the points of $d \cap k_n$ ordered along d, the arc $[x_1, x_{2p}] \subset d$ joining x_1 to x_{2p} and not containing x_2 does not lie in τ. For each $i \leq p$, add to k_n the arc of d joining x_i to x_{2p+1-i} and containing $[x_1, x_{2p}]$ (cf. figure 3).

Repeating this operation for each connected component of $\partial'\bar{S}(\lambda^i)$ intersecting k_n yields a family k_n' of simple closed curves. Each component of k_n' is either homotopic to a simple closed geodesic or homotopic to a point. Let us denote by $|\gamma_n^i|$ the union of the closed geodesics thus obtained, at this point we do not know whether or not $|\gamma_n^i|$ is empty. Associating to each leaf of $|\gamma_n^i|$ the half-weight of ν_n yields a weighted multi-curve $\gamma_n^i \in \mathscr{ML}(S(\lambda^i))$ such that $l_{s_n}(\gamma_n^i) \leq l_{s_n}(\nu_n) + i(\nu_n, \partial'\bar{S}(\lambda^i))l_{s_n}(\partial'\bar{S}(\lambda^i)) \leq l_{s_n}(\lambda_n) + i(\lambda_n, \partial'\bar{S}(\lambda^i))l_{s_n}(\partial'\bar{S}(\lambda^i))$ (the last inequality comes from our assumptions on $l_{s_n}(\nu_n)$ and on $i(\nu_n, \partial'\bar{S}(\lambda^i))$).

Claim 3.9. *No component of $k_n' - A$ can be homotoped into A.*

Proof. Fix n and let κ be a component of $k_n' - A$. If κ is a component of $k_n - A$, then it is a geodesic segment. Since A is also a geodesic segment, κ can not be homotoped into A. If κ is not a component of $k_n - A$, it is the union of two components κ_1 and κ_2 of $k_n - A$ and of an arc κ_3 of $\partial'\bar{S}(\lambda^i)$ not lying in τ. If there is an arc $\kappa_4 \subset \partial'\bar{S}(\lambda^i) - (\kappa_1 \cup \kappa_2)$ such that $\kappa_1 \cup \kappa_4 \cup \kappa_2$ can be homotoped into A, then κ_1 and κ_2 are homotopic relative to $A \cup \partial'\bar{S}(\lambda^i)$. So κ_1 and κ_2 lie in the same branch b of τ and κ_4 lies in b. By construction $\kappa_3 \not\subset b$ hence $\kappa_1 \cup \kappa_3 \cup \kappa_2$ can not be homotoped into A. The claim is thus proved. $\square$

Claim 3.10. *There exists a simple closed curve $c' \subset S(\lambda^i)$ such that $i(\gamma_n^i, c') \geq \frac{1}{2}\int_A d\lambda^i$ for large n.*

Proof. As we have already seen in the preceding proof, any connected component of $k'_n - A$ is either homotopic to a rail of τ or to the union of two rails and of a segment of $\partial' \bar{S}(\lambda^i)$. There are

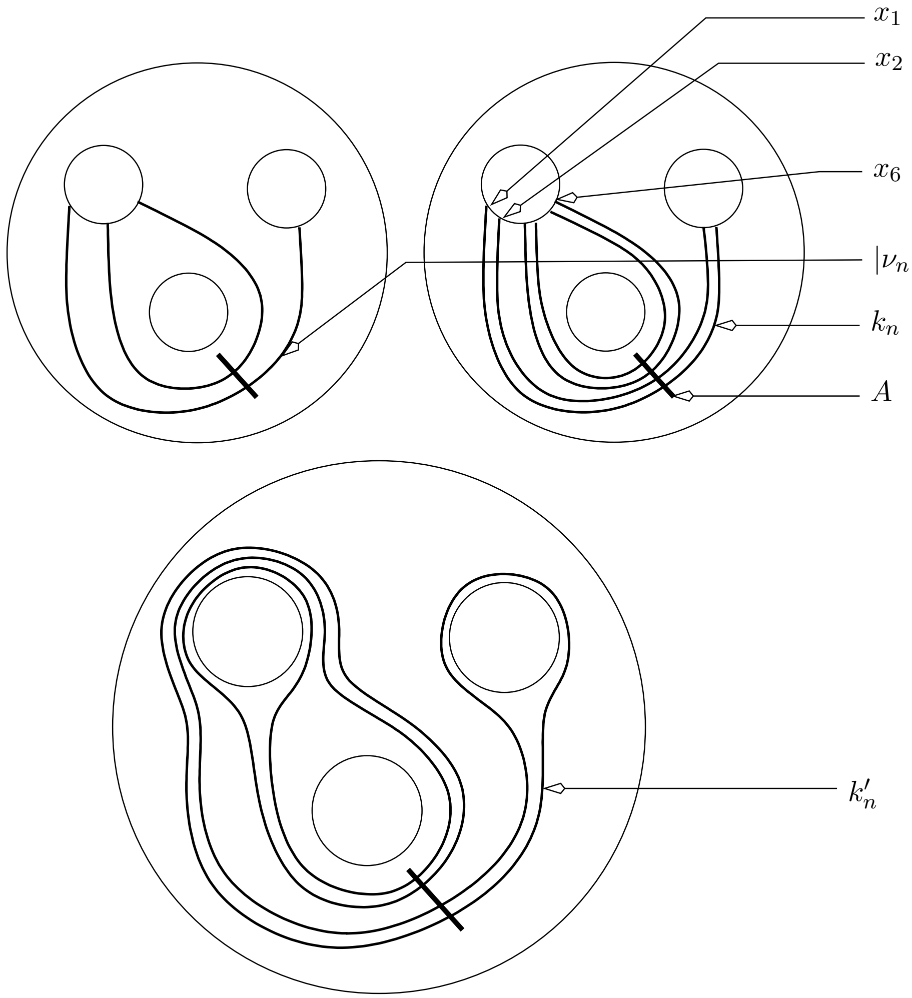

Figure 3: An example for the construction of k'_n

only finitely many possibilities for the homotopy class relative to $A \cup \partial' \bar{S}(\lambda^i)$ of such an arc. On the other hand, there are infinitely many simple closed geodesics which pass near A. Therefore we can find a curve c' and an arc $A' \subset c'$ which is close to A so that no component of $k'_n - A'$ can be homotoped in $c' - A'$. By Claim 3.9, no component of $k'_n - A'$ can be homotoped into A', hence no component of $k'_n - A'$ can be homotoped into c'. It follows that we have $i(\gamma^i_n, c) \geq \int_{A'} d\nu^i_n$. Since A lies in the surface embraced by λ^i, it does not intersect any closed leaf of $\dot{\lambda}$. Therefore, by Claim 2.3, $\int_{A'} d\nu^i_n$ converges to $\int_{A'} d\lambda^i$. Since A' is close to A, we have $\int_{A'} d\lambda^i = \int_A d\lambda^i$ and $i(\gamma^i_n, c') \geq \frac{1}{2} \int_A d\lambda^i$ for large n. □

Since λ^i intersects A transversely, we have $\int_A d\lambda^i > 0$. It follows from the claim above that, for large n, $i(\gamma^i_n, c') > 0$, therefore γ^i_n is not trivial.

Let $d \subset S(\lambda^i)$ be a simple closed curve. By Claim 2.3, $i(d, \nu_n) \longrightarrow i(d, \lambda)$. By construction $i(d, \gamma^i_n) \leq i(d, \nu_n)$, hence, up to extracting a subsequence, γ^i_n converges in $\mathscr{ML}(\partial M)$ to a measured geodesic lamination γ^i_∞. Since we have $i(\gamma^i_n, c') \geq \int_A d\nu^i_n \longrightarrow \int_A d\lambda^i > 0$, the measured geodesic lamination γ^i_∞ is not trivial.

Let us consider a subsequence such that $|\gamma^i_n|$ converges to a geodesic lamination L^i in the Hausdorff topology. Let us show that L^i is an expansion of λ^i. Since $|\gamma^i_\infty|$ lies in $L^i \subset S(\lambda^i)$, this is equivalent to the following claim.

Claim 3.11. *The geodesic lamination $|\gamma^i_\infty|$ does not intersect λ^i transversely.*

Proof. Let $w(\nu_n)$ be the weight of ν_n. Since each leaf of γ_n is homotopic to a component of k'_n and has a weight equal to $w(\nu_n)$, we have $i(\gamma_n,\nu_n)\leq w(\nu_n)\int_{k'_n}d\nu_n$. Since k_n is the union of two parallel copies of $|\nu_n|\cap S(\lambda^i)$, we have $\int_{k_n}d\nu_n=0$. To obtain k'_n, we have added to k_n some arcs lying in $\partial'\bar{S}(\lambda^i)$. Since the intersection number of each of these arc with ν_n is less than $i(\partial'\bar{S}(\lambda^i),\nu_n)$, we have $\int_{k'_n}d\nu_n\leq\sharp(|\nu_n|\cap\partial'\bar{S}(\lambda^i))i(\partial'\bar{S}(\lambda^i),\nu_n)$. Thus we have $i(\gamma_n,\nu_n)\leq w(\nu_n)\sharp(|\nu_n|\cap\partial'\bar{S}(\lambda^i))i(\partial'\bar{S}(\lambda^i),\nu_n)=i(\partial'\bar{S}(\lambda^i),\nu_n)^2$. By Claim 2.3, $i(\partial'\bar{S}(\lambda^i),\nu_n)\longrightarrow 0$, hence $i(\gamma^i_n,\nu_n)\longrightarrow 0$.

It is easy now to conclude that $|\gamma^i_\infty|$ does not intersect λ^i transversely. □

Doing this construction for each irrational component of λ and adding the result to the closed leaves of λ yields a geodesic lamination L expanding λ which satisfies the conditions of Lemma 3.5.

4 Properness of $b_{\mathscr{C}\mathscr{C}}$ and $b_{\mathscr{R}}$

Let M be a compact orientable hyperbolic 3-manifold, let (σ_n) be a sequence of geometrically finite metrics on $int(M)$ and let $\lambda_n\subset\mathscr{ML}(\partial M)$ be the bending measured geodesic lamination of σ_n. In the previous section, we have shown that if $\dot{\lambda}_n$ converges to $\dot{\lambda}$ in $\mathscr{P}^+(M)/\mathscr{R}$, then there is a sequence $\rho_n:\pi_1(M)\to Isom(\mathbb{H}^3)$ of representations associated to (σ_n) such that a subsequence of (ρ_n) tends to a representation $\rho_\infty:\pi_1(M)\to Isom(\mathbb{H}^3)$. By [Jor], ρ_∞ is discrete and faithful. Next, we are going to show that ρ_∞ is geometrically finite and corresponds to a metric σ_∞ on the interior of M. Finally we will show that a subsequence of (σ_n) converges to σ_∞ and conclude that $b_{\mathscr{R}}$ is proper.

Let us first keep apart the closed curves of ∂M that have lengths tending to 0, in the same way as in [BoO].

Lemma 4.1. *Up to extracting a subsequence, there is a multi-curve $L^0\subset\partial_{\chi<0}M$ with the two following properties :*

- *Any connected component c of L^0 satisfies $l_{s_n}(c)\longrightarrow 0$;*
- *If $c\subset\partial_{\chi<0}M$ is a simple closed curve which is not in the homotopy class of any connected component of L^0, then the sequence $l_{s_n}(c)$ is bounded from below by a strictly positive number (depending on c).*

Proof. See [BoO, Lemme 15]. □

It will follow from the continuity of $b_{\mathscr{R}}$ ([Le3]) that L^0 is in fact the union of the closed leaves of λ' with a weight equal to π. For the moment, let us just show that L^0 does not intersect $|\dot{\lambda}|$ transversely.

Claim 4.2. *The multi-curve L^0 does not intersect $|\dot{\lambda}|$ transversely.*

Proof. We will follow [BoO, lemme 16]. Let c be a connected component of L^0 and let λ' be the representative of $\dot{\lambda}$ that lies in $\mathscr{P}(M)$. If c intersects $|\dot{\lambda}|$ transversely then for large n, we have $i(c,\lambda_n)\geq\frac{1}{2}i(c,\lambda')$ (cf. Claim 2.3). It follows that c is not homotopic to a leaf of λ_n, and in particular c is not homotopic to a cusp of s_n. Therefore c is homotopic to a geodesic of s_n and, up to extracting a subsequence, we have $l_{s_n}(c)\to 0$. Therefore c is the core of a Margulis tube of ∂M and c is far from the boundary of this tube. Since $i(c,\lambda_n)\geq\frac{1}{2}i(c,\lambda')>0$, this implies that $l_{s_n}(\lambda_n)$ tends to ∞, contradicting Lemma 3.6. □

Let c be component of L^0. Since c does not intersect $\dot\lambda \in \mathscr{P}^+(M)/\mathscr{R}$, c represents a non-trivial element $i_*(c)$ of $\pi_1(M)$. Since ρ_n converges to ρ_∞ and $l_{\sigma_n}(c) \leq l_{s_n}(c) \to 0$, it follows that $\rho_\infty \circ i_*(c)$ is a parabolic isometry.

4.1 The metrics in the boundary

Since $\dot\lambda_n$ tends to $\dot\lambda$, up to extracting a subsequence, the closed leaves of λ_n with a weight equal to π do not depend on n. Let us denote their union by $\lambda^{(p)} \subset L^0$. Let $h_n : M - \lambda^{(p)} \to N(\rho_n)$ be a homeomorphism associated to σ_n. The map $f_n = h_n \circ i : \partial_{\chi<0}M - \lambda^{(p)} \to \mathbb{H}^3/\rho_n(\pi_1(M))$ is a convex pleated surface whose bending measured geodesic lamination is $\lambda_n - \lambda^{(p)}$. We will show that this sequence of pleated surfaces tends to a pleated surface $f_\infty : \partial M - L^0$. As in [BoO], we will then see that f_∞ is a connected component of $\partial N(\rho_\infty)$ and use a homological argument to conclude that ρ_∞ is geometrically finite.

Let S be a connected component of $\partial_{\chi<0}M - L^0$ and let $\hat S$ be the universal cover of S. Given an isometry from $\hat S$ endowed with the metric lifted from s_n to a subset of $\mathbb{H}^2$, the covering transformations yields a discrete faithful representation $r_n : \pi_1(S) \to Isom(\mathbb{H}^2)$ associated to s_n. We say that the restriction s_n^S of s_n to S converges algebraically if we can choose representations r_n associated to the s_n^S which converge algebraically.

Lemma 4.3. *The sequence (s_n^S) contains an algebraically converging subsequence.*

Proof. Let us assume the contrary. By the definition of L^0, $l_{s_n}(\partial\bar S) \longrightarrow 0$. So we can use the theory of [MoS1]. By [MoS1] and [MoO] a subsequence of the sequence (s_n^S) tends to a Λ-measured geodesic lamination $\mathscr{L}$. Since $l_{s_n}(\lambda_n)$ is bounded, by Theorem A.1, $|\dot\lambda|$ does not intersects $|\mathscr{L}|$ transversely.

Claim 4.4. *The support of $\mathscr{L}$ is a multi-curve.*

Proof. Assume first that $|\mathscr{L}|$ contains an irrational lamination. Let $\eta > 0$, since $|\dot\lambda|$ does not intersect $|\mathscr{L}|$ transversely, there is a simple closed curve c_η such that $i(\lambda', c_\eta) \leq \eta$ and $\delta_{\mathscr{L}}(c_\eta) > 0$. By Claim 2.3, for large n, we have $i(\lambda_n, c_\eta) \leq 2\eta$. Since $\delta_{\mathscr{L}}(c_\eta) > 0$, then $l_{s_n}(c_\eta) \longrightarrow \infty$. By the slight bending Lemma, $\exists C_\eta$ such that $l_{s_n}(c_\eta) \leq C_\eta l_{\sigma_n}(c_\eta^*)$. Since σ_n converges algebraically, $l_{\sigma_n}(c_\eta^*)$ is bounded and the case studied here does not occur, namely $\mathscr{L}$ does not contain any irrational lamination.

Let c be a closed leaf of $|\mathscr{L}|$ such that $\delta_{\mathscr{L}}(c) > 0$. By [MoS1], we have $l_{s_n}(c) \longrightarrow \infty$. Since $|\dot\lambda|$ does not intersect $|\mathscr{L}|$ transversely, we have $i(c, \lambda') = 0$. By Claim 2.3, $i(c, \lambda_n)$ tends to 0. Therefore by the slight bending Lemma ([Le1, Lemma A.1]), there exists C such that for large n, we have $l_{s_n}(c) \leq C l_{\sigma_n}(c^*)$. This yields the same contradiction as in the preceding paragraph. Thus for any closed leaf c of $\mathscr{L}$, we have $\delta_{\mathscr{L}}(c) = 0$.

By Claim 2.1, it follows that $|\mathscr{L}|$ is a multi-curve. □

By the construction of the Λ-tree $\mathscr{T}_{\mathscr{L}}$, the sequence s_n^S is bounded when restricted to any component of $S - |\mathscr{L}|$. Therefore there is $C_1 > 0$ such that for any n we have $inf\{l_{s_n}(c)/\ c \text{ is a closed geodesic of } S - |\mathscr{L}|\} > C_1$. By the definition of L^0, there is $C_2 > 0$ such that we have $l_{s_n}(c) > C_2$ for any n and for any component c of $|\mathscr{L}|$. Since $|\mathscr{L}|$ is a multi-curve, we have $\delta_{\mathscr{L}}(|\mathscr{L}|) = 0$. Hence $l_{s_n}(\mathscr{L})$ is bounded from above. It follows that $\exists C_3 > 0$ such that, for any n, $inf\{l_{s_n}(c)/\ c \text{ intersects } \mathscr{L} \text{ transversely}\} > C_3$. Now we have $inf\{l_{s_n}(c)/\ c \text{ is a closed geodesic of } S\} > \min\{C_1, C_2, C_3\} > 0$. Therefore, by Mumford's Lemma (cf. [CEG, Proposition 3.2.13]), the sequence s_n^S is bounded in the modular space, namely there are diffeomorphisms ψ_n such that the sequence $(\psi_{n*}s_n^S)$ is bounded in the algebraic topology.

Since the restriction of s_n^S to each component of $S - |\mathscr{L}|$ is bounded, we may assume that ψ_n is the composition of Dehn twists along the leaves of $|\mathscr{L}|$.

Claim 4.5. *The number of turns of the Dehn twists composing ψ_n is bounded.*

Proof. For a closed curve $c \in \partial M$ and $n \in \mathbb{N}\cup\{\infty\}$ we will denote by $\rho_n(c)$ the corresponding element of $\rho_n(\pi_1(M))$. The isometry $\rho_n(c)$ depends only on the free homotopy class of $c \in \partial M$. Let c be a connected component of $\mathscr{L}$. Since (ρ_n) converges algebraically, $\rho_n(c)$ converges to $\rho_\infty(c)$. Let us prove that $\rho_\infty(c)$ is not parabolic. Assume the contrary, then we have $l_{\sigma_n}(c^*) \longrightarrow 0$. Since $i(\lambda_n, c) \longrightarrow 0$, it follows from the slight bending Lemma that we have $l_{s_n}(c) \longrightarrow 0$. Hence c lies in L^0 contradicting the fact that $|\mathscr{L}|$ lies in $S \subset \partial M - L^0$. So $\rho_\infty(c)$ is hyperbolic and up to extracting a subsequence, $\rho_n(c)$ is also hyperbolic.

Let us assume that ψ_n turns p_n times toward c with $p_n \longrightarrow \infty$. Let F^1 and F^2 be the connected components (which may coincide) of $S - |\mathscr{L}|$ whose boundary contains c. Since $|\mathscr{L}|$ does not intersect λ' transversely, for any $\varepsilon > 0$, F^j (with $j = 1, 2$) contains a simple closed curve $c_{j,\varepsilon}$ with $i(c_{j,\varepsilon}, \lambda') \leq \varepsilon$. We have $\lambda' \in \mathscr{P}(M)$, let us fix $\varepsilon < \min\{2\pi, \eta\}$ (here η comes from condition $b)$). The condition $c)$ implies that $i_*(c_{j,\varepsilon}) \in \pi_1(M)$ is not trivial. Fix a point $x \in F^1$ and consider a loop $a \ni x$ homotopic to $c_{1,\varepsilon}$ and a loop $b \ni x$ homotopic to $c_{2,\varepsilon}$. We are in the situation of figure 4.

Since $\psi_{n*}s_n^S$ is bounded, $l_{s_n^S}(\psi_n(ab))$ is bounded and $l_{\sigma_n}(\psi_n(ab)^*)$ is also bounded. Therefore

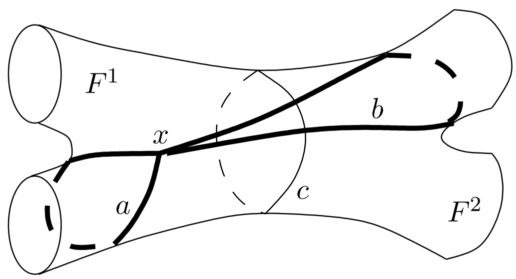

Figure 4: Controlling the number of Dehn twists

we can extract a subsequence such that $\rho_n(\psi_n(ab)) = \rho_n(ac^{p_n}bc^{-p_n})$ converges. Since $\rho_n(a)$ converges, it follows that $\rho_n(c^{p_n}bc^{-p_n}) = \rho_n(c)^{p_n}\rho_n(b)\rho_n(c)^{-p_n}$ contains a converging subsequence.

Let $l_n \subset \mathbb{H}^3$ be the axis of $\rho_n(c)$, let $x_n \in l_n$ be a sequence converging to $x_\infty \in l_\infty$ and let us extract a subsequence such that $\rho_n(c)^{p_n}\rho_n(b)\rho_n(c)^{-p_n}(x_n)$ converges. Since $\rho_n(c)$ tends to $\rho_\infty(c)$ which is hyperbolic, $\rho_n(c)^{-p_n}(x_n)$ tends to a fixed point of $\rho_n(c)$. Now, we know that $\rho_n(b)$ tends to $\rho_\infty(b)$, that $\rho_n(c)^{-p_n}(x_n)$ tends to the boundary at infinity and that $d_{\mathbb{H}^3}(\rho_n(c)^{-p_n}(x_n), \rho_n(b)\rho_n(c)^{-p_n}(x_n)) = d_{\mathbb{H}^3}(x_n, \rho_n(c)^{p_n}\rho_n(b)\rho_n(c)^{-p_n}(x_n))$ is bounded. This implies that $\rho_n(c)^{-p_n}(x_n)$ tends to a fixed point of $\rho_\infty(b)$. Therefore $\rho_\infty(b)$ and $\rho_\infty(c)$ have a common fixed point. Since $\rho_\infty(\pi_1(M))$ is a discrete group, $\rho_\infty(b)$ and $\rho_\infty(c)$ are multiples of the same indivisible element of $\rho_\infty(\pi_1(M))$, namely there exist $d \in \pi_1(M)$ and $p, q \in \mathbb{N}$ with $b = d^p$ and $c = d^q$. Fix n and choose a homeomorphism $(\tilde{M}, \sigma_n) \rightarrow \mathbb{H}^3 \cup \Omega(\rho_n)$. Recall that $c_{2,\varepsilon}$ is freely homotopic to b. Let $\tilde{c}$ and $\tilde{c}_{2,\varepsilon} \in \Omega(\rho_n)$ be lifts of c and $c_{2,\varepsilon}$, respectively, which are left invariant by $\rho_n(d^{pq})$. The family $\{\tilde{c}, \rho_n(d)(\tilde{c}), \ldots, \rho_n(d^{q-1})(\tilde{c})\}$ cuts $\partial\overline{\mathbb{H}}^3 \approx S^2$ into q crescents $Z_1, \ldots, Z_q$. Since $c_{2,\varepsilon}$ does not intersect c, for any $i \in \mathbb{Z}$, $\rho_n(d^i)(\tilde{c}_{2,\varepsilon})$ lies in one Z_j. Furthermore $Z_j = \rho_n(d)(Z_{j-1})$, hence the number of translates of $\tilde{c}_{2,\varepsilon}$ that lie in Z_j does not depend on j. It follows that q divides p. By exchanging the roles of c and $\tilde{c}_{2,\varepsilon}$, we get that p divides q. Thus we get $q = p$. So c and $\tilde{c}_{2,\varepsilon}$ are homotopic. It follows from [ShW] that $c_{2,\varepsilon} \cup b$ bounds an essential annulus. Since $i(\lambda', c \cup c_\varepsilon^2) \leq \varepsilon < \eta$, this contradicts condition $c)$.

□

Now we can conclude that, up to extracting a subsequence, $\exists n_1$ such that the diffeomorphisms $\psi_n \circ \psi_{n_1}^{-1}$ are isotopic to the identity. This concludes the proof of Lemma 4.3. □

Using the convergence of the metrics on the boundary, we are going to construct a convex pleated surface $f_\infty : S \to \mathbb{H}^3/\rho_\infty(\pi_1(M))$.

Lemma 4.6. *Let S be a connected component of $\partial M - L^0$. Then there exists a convex pleated surface $f_\infty : S \to \mathbb{H}^3/\rho_\infty(\pi_1(M))$ which is homotopic to the inclusion map.*

Proof. Let $r_n : \pi_1(S) \to Isom(\mathbb{H}^2)$ be a representation associated to s_n such that (r_n) converges algebraically to a representation $r_\infty : \pi_1(S) \to Isom(\mathbb{H}^2)$.

To extract from (f_n) a converging subsequence, we will follow the ideas of [CEG]. For this, we need to show that the $f_n(S)$ all intersect the image in $\mathbb{H}^3/\rho_n(\pi_1(M))$ of a compact subset of $\mathbb{H}^3$. Let x_n be the image in $\mathbb{H}^3/\rho_n(\pi_1(M))$ of the base point $o \in \mathbb{H}^3$, then we have :

Lemma 4.7 ([BoO], lemma 17). *There is a sequence of arcs k_n joining $f_n(S)$ to the base point x_n with uniformly bounded length.*

Let S' be the component of $\partial M - \lambda^{(p)}$ that contains S and let y_n be the vertex of k_n that is not x_n. Using Lemmas 4.3 and 4.7 and the arguments of [CEG, §5.2], we get that, up to extracting a subsequence, the sequence of pleated surfaces $f_n : S' \to \mathbb{H}^3/\rho_n(\pi_1(M))$ with basepoint y_n converges to a pleated surface $f_\infty : S \to \mathbb{H}^3/\rho_\infty(\pi_1(M))$. Since all the f_n are homotopic to the inclusion map, their limit f_∞ is also homotopic to the inclusion map. □

By [BoO, Lemme 21] (see also [Le3, Lemma 3.1]), either $f_\infty(S)$ is a convex surface, namely there is a convex set $C_{f_\infty} \subset \mathbb{H}^3/\rho_\infty(\pi_1(M))$ whose boundary is $f_\infty(S)$, or $f_\infty(S)$ lies in a totally geodesic surface $\mathbb{H}^3/\rho_\infty(\pi_1(M))$. It follows also from the proof of [BoO, Lemme 21] (see also [Le3, Lemma 3.1]) that the convex core $N(\rho_\infty)$ of $\rho_\infty(\pi_1(M))$ lies either in $C_{\hat{f}_\infty}$ (when $f_\infty(S)$ is convex) or in $f_\infty(S)$ (in the other case). If $f_\infty(S)$ is not a convex surface (as defined above), then $\rho_\infty(\pi_1(M))$ is a fuchsian group and the restriction of f_∞ to $S - |\dot\lambda_\infty|$ is a two-sheeted covering of the interior of $f_\infty(S)$ considered as a two dimensional surface with boundary (see [Le3, Lemma 3.8] for more details). It follows that M is an interval bundle over a closed surface $I \times F$ and that $|\dot\lambda_\infty|$ is a section of the bundle $\partial S \times I$. This contradicts the fact that $\dot\lambda_\infty$ lies in $\mathscr{P}^+(M)/\mathscr{R}$. Therefore $f_\infty(S)$ is a convex surface and f_∞ is a covering onto its image (cf. [EpM]). It follows from the same arguments that f_∞ is a one-sheeted covering, namely a homeomorphism into its image (see [BoO, Lemme 21]).

Since f_∞ is a pleated surface, we have $f_\infty(S) \subset N(\rho_\infty)$. Since $N(\rho_\infty) \subset C_{f_\infty}$, the surface $f_\infty(S)$ is a component of the boundary of $N(\rho_\infty)$.

Let us do the same construction for each component of $\partial_{\chi<0}M - L_0$ and denote by $f_\infty : \partial_{\chi<0}M - L_0 \to \partial N(\rho_\infty)$ the resulting map. By the above, the image of f_∞ is a union of components of $\partial N(\rho_\infty)$ and f_∞ is a local homeomorphism.

Claim 4.8. *The map $f_\infty : \partial_{\chi<0}M - L^0 \to \partial N(\rho_\infty)$ is injective.*

Proof. Assume the contrary. We have shown above that the restriction of f_∞ to any component of $\partial_{\chi<0}M - L_0$ is a homeomorphism into its image. Hence there are two connected components S and S' of $\partial_{\chi<0}M - L_0$ such that $f_\infty(S) \cap f_\infty(S') \neq \emptyset$. Since $f_\infty(S)$ and $f_\infty(S')$ are connected components of $\partial N(\rho_\infty)$, they intersect if and only if they are equal. Since $N(\rho_n)$ lies between $f_n(S)$ and $f_n(S')$, this is possible only if $f_\infty(S)$ and $f_\infty(S')$ are totally geodesic and if $\rho_\infty(\pi_1(M))$ is fuchsian (see [BoO, Lemma 21]). In this case, M is homeomorphic to $S \times I$ and $|\dot\lambda_\infty|$ lies in a section of the bundle $\partial S \times I$ contradicting again the fact that $\dot\lambda_\infty$ lies in $\mathscr{P}^+(M)/\mathscr{R}$. □

We can now prove, following [BoO], that ρ_∞ is geometrically finite and is associated to a complete hyperbolic on $int(M)$.

Lemma 4.9. *There is a geometrically finite metric σ_∞ on int(M) such that ρ_∞ is a representation associated to σ_∞.*

Proof. To each leaf c of L^0 correspond two cusps of $\partial_{\chi<0}M - L^0$. The images of these two cusps under f_∞ are two totally geodesic open annuli tending to a rank one cusp of $\mathbb{H}^3/\rho_\infty(\pi_1(M))$. Remove these two annuli from $f_\infty(\partial_{\chi<0} - L^0)$ and join the two boundary components of the remaining surface by a compact annulus. Doing this for each leaf of L^0, we get a compact surface $F_\infty \subset \mathbb{H}^3/\rho_\infty(\pi_1(M))$. The same construction in M yields a surface S_∞ such that $S_\infty \cup \partial_{\chi=0}M$ bounds a compact subset of M. Let us denote by F'_∞ the surface obtained by adding to F_∞ the boundary of a neighborhood of the rank 2 cusps of $\mathbb{H}^3/\rho_\infty(\pi_1(M))$. From f_∞, we get easily a homeomorphism $f : S_\infty \cup \partial_{\chi=0}M \to F'_\infty$ which is homotopic to the inclusion. Since $S_\infty \cup \partial_{\chi=0}M$ bounds a compact set in M, F'_∞ is homologous to 0 and therefore bounds a finite volume cycle. Adding some finite volume sets (which are rank 2 cusps and "slices" of rank 1 cusps) to this cycle, we get a new cycle C_∞ which is bounded by $f_\infty(\partial_{\chi<0}M - L^0)$ and has finite volume.

By Claim 4.8, f_∞ is a homeomorphism into the union of some components of $\partial N(\rho_\infty)$. If some component of $\partial N(\rho_\infty)$ did not lie in $f_\infty(\partial_{\chi<0}M - L^0)$ then the corresponding component b of $\mathbb{H}^3/\rho_\infty(\pi_1(M)) - N(\rho_\infty)$ would lie in C_∞. But, since the projection $\mathbb{H}^3/\rho_\infty(\pi_1(M)) - N(\rho_\infty) \longrightarrow \partial N(\rho_\infty)$ does not increase distances, b has infinite volume. So b does not lie in C_∞. It follows that we have $f_\infty(\partial_{\chi<0}M - L^0) = \partial N(\rho_\infty)$ and that ρ_∞ is geometrically finite.

Since the homeomorphism $f_\infty : \partial_{\chi<0}M - L^0 \to \partial N(\rho_\infty)$ is homotopic to the identity, there is a homotopy equivalence $M - (L^0 \cup \partial_{\chi=0}M) \to N(\rho_\infty)$ which agrees with f_∞ on $\partial_{\chi<0}M - L^0$. By [Wa], this homotopy equivalence can be realized by a homeomorphism. Composing this homeomorphism with the natural homeomorphism $int(N(\rho_\infty)) \to \mathbb{H}^3/\rho_\infty(\pi_1(M))$ yields the desired geometrically finite metric σ_∞ on $int(M)$. □

4.2 Convergence of (σ_n) to σ_∞

To conclude that $b_{\mathscr{R}}$ is proper, it remains to show that a subsequence of (σ_n) converges to σ_∞. First, we are going to show that, up to extracting a subsequence, $\rho_n(\pi_1(M))$ converges strongly to $\rho_\infty(\pi_1(M))$. Let $\Omega(\rho_n)$ be the domain of discontinuity of $\rho_n(\pi_1(M))$. We will use a result of [JoM] (see also [MaT]):

Theorem 4.10. *Assume that $\Omega(\rho_\infty)$ is not empty. If $(\Omega(\rho_n))$ converges to $\Omega(\rho_\infty)$ in the sense of Carathéodory, then ρ_n converges strongly to ρ_∞.*

Let us recall that $(\Omega(\rho_n))$ converges to $\Omega(\rho_\infty)$ in the sense of Carathéodory if and only if $(\Omega(\rho_n))$ satisfies the two following conditions :

- every compact subset $K \subset \Omega(\rho_\infty)$ lies in $\Omega(\rho_n)$ for all sufficiently large n;

- every open set O which lies in $\Omega(\rho_n)$ for infinitely many n also lies in $\Omega(\rho_\infty)$.

We are going to show that $\Omega(\rho_n)$ and $\Omega(\rho_\infty)$ satisfy these two conditions.

Lemma 4.11. *Under the hypothesis of Proposition 3.1, a subsequence of (ρ_n) converges strongly to ρ_∞.*

Proof. Since ρ_∞ is geometrically finite, $\Omega(\rho_\infty)$ is not empty.

Let $K \subset \Omega(\rho_\infty)$ be a compact connected subset and let $\Omega(\rho_\infty)_1$ be the connected component of $\Omega(\rho_\infty)$ that contains K. Let $\tilde{\Sigma}_1 \subset \partial\tilde{N}(\rho_\infty)$ and $K' \subset \tilde{\Sigma}_1$ be the images of $\Omega(\rho_\infty)_1$ and of K, respectively, under the retraction map $\Omega(\rho_\infty)_1 \to \partial C(\rho_\infty)$. Let $\hat{f}_n : \mathbb{H}^2 \to \mathbb{H}^3$ be a lift of f_n and let $\hat{\lambda}_n$ be the bending geodesic measured lamination of $\hat{f}_n$. Let us extract a subsequence such that $|\hat{\lambda}_n|$ converges for the Hausdorff topology to a geodesic lamination $\hat{L} \subset \mathbb{H}^2$. Let $\hat{x}_1, ..., \hat{x}_\infty \in \mathbb{H}^2 - \hat{L}$ be a sequence of points such that any connected component of $\mathbb{H}^2 - \hat{L}$ contains one of these points and let Π_i be the support plane at $\hat{f}_\infty(\hat{x}_i)$. The boundary of Π_i in $\partial\overline{\mathbb{H}}^3$ is a circle. This circle bounds a disc $D_i \subset \partial\overline{\mathbb{H}}^3$ such that $int(D_i) \subset \Omega(\rho_\infty)$. We have $\Omega(\rho_\infty)_1 = \bigcup_i int(D_i)$. Let $i \in \mathbb{N}$; for n large enough, $\hat{x}_i$ lies in $\mathbb{H}^2 - |\hat{\lambda}_n|$. Let $\Pi_{i,n}$ be the support plane at $\hat{f}_n(\hat{x}_i)$. The boundary of $\Pi_{i,n}$ in $\partial\overline{\mathbb{H}}^3$ bounds a disc $D_{i,n} \subset \partial\overline{\mathbb{H}}^3$ such that $int(D_{i,n}) \subset \Omega(\rho_n)$. We have $\bigcup_i int(D_{i,n}) \subset \Omega(\rho_n)$.

Since $(\hat{f}_n)$ converges to $\hat{f}_\infty$, it follows that, for any $i \in \mathbb{N}$, $(D_{i,n})$ converges to D_i in the Hausdorff topology. Since K is a compact set lying in $\Omega(\rho_\infty)$, it is covered by finitely many compact sets K_j such that each K_j lies in the interior of some D_i. For n large enough, each K_j lies in the interior of the corresponding disc $D_{i,n}$, hence K lies in $\Omega(\rho_n)$.

Let O be an open set lying in $\Omega(\rho_n)$ for infinitely many n and let $w \in O \cap L_{\rho_\infty}$. Since the fixed points of hyperbolic isometries are dense in L_{ρ_∞} and since (ρ_n) converges to ρ_∞, there is a sequence of points (w_n) converging to w such that $w_n \in L_{\rho_n}$. For n large enough, we have $w_n \subset O \cap L_{\rho_n}$, contradicting our hypothesis that $O \subset \Omega(\rho_n)$ for infinitely many n. □

Let $n \in \overline{\mathbb{N}}$, let $M_n = \mathbb{H}^3/\rho_n(\pi_1(M)) \approx int(M)$, let $x_n \in M_n$ be the projection of the origin $o \in \mathbb{H}^3$ and let $h_n : M - (\lambda_n^{(p)} \cup \partial_{\chi=0}M) \to N(\rho_n)$ be a homeomorphism associated to σ_n.

Since $\rho_n(\pi_1(M))$ converges strongly to $\rho_\infty(\pi_1(M))$, there are sequences $k_n \longrightarrow 1$ and $r_n \longrightarrow \infty$ and a sequence of maps $\tilde{u}_n : B(o, r_n) \subset \mathbb{H}^3 \to \mathbb{H}^3$ with the following properties (cf. [BeP] for example) :

- the map $\tilde{u}_n$ is a k_n-bilipschitz diffeomorphism onto its image, $u_n(o) = o$ and $\tilde{u}_n$ converges to the identity map on any compact set;

- the map $\tilde{u}_n$ is equivariant, namely for any point $x \in B(o, r_n)$ and any element $a \in \pi_1(M)$ such that $\rho_\infty(a)(x) \in B(o, r_n)$, we have $u_n \circ \rho_\infty(a)(x) = \rho_n(a) \circ u_n(x)$.

It follows from this second property that $\tilde{u}_n$ yields a quotient map $u_n : B(x_\infty, r_n) \subset M_\infty \to M_n$ which is also a k_n-bilipschitz diffeomorphism onto its image.

Since M_∞ is geometrically finite, for r_n large enough, $B(x_\infty, r_n) \subset M_\infty$ is homeomorphic to M and the inclusion map $B(x_\infty, r_n) \subset M_\infty \approx int(M)$ is a homotopy equivalence. This implies that $u_n(B(x_\infty, r_n))$ is homeomorphic to M. Since $\tilde{u}_n$ is equivariant, the inclusion map $u_n(B(x_\infty, r_n)) \subset M_n \approx int(M)$ is a homotopy equivalence. By a result of [MMS], $M - B(x_\infty, r_n)$ and $M - u_n(B(x_\infty, r_n))$ are homeomorphic to $\partial M \times (0, 1]$ and we can extend the map
$u_n : B(x_\infty, r_n) \subset int(M) \to int(M)$ to a homeomorphism $v_n : M \to M$ homotopic to the identity. To show that (σ_n) converges to σ_∞, it remains to show that v_n is isotopic to the identity for large n.

Let $n \in \mathbb{N}$, the map $f_n : \partial_{\chi<0}M - L^0 \to \partial N(\rho_n) \subset M_n \approx int(M)$ is the restriction of an homeomorphism h_n associated to ρ_n. Therefore f_n is isotopic to the inclusion map $\partial_{\chi<0}M - L^0 \to M$. Since u_n is a k_n-bilipschitz map, for n large enough, $B(x_n, r_n - 1)$ lies in $u_n(B(x_\infty, r_n))$. Let S be a connected component of $\partial_{\chi<0}M - L^0$. By Lemma 4.3 and 4.7, for n large enough, the thick part of $f_n(S)$ lies in $B(x_n, r_n - 1) \subset u_n(B(x_\infty, r_n))$. In the proof of Lemma 4.6 we have shown that $f_n : S \to \partial N(\rho_n)$ converges to $f_\infty : S \to \partial N(\rho_\infty)$ and we already know that $\tilde{u}_n$ converges to the identity on any compact set. It follows that $u_n^{-1} \circ f_n = v_n^{-1} \circ f_n$ converges to f_∞ on any compact set. Hence, for n large enough, the restrictions of $v_n^{-1} \circ f_n$ and of f_∞ to S are isotopic. Since f_n and f_∞ are isotopic to the inclusion map $S \to M$, the restriction of v_n to $S \subset \partial M$ is isotopic to the inclusion map $S \to M$.

Let $\mathscr{V}(L^0)$ be an annular neighborhood of L^0. Since $\dot{\lambda} \in \mathscr{P}^+(M)/\mathscr{R}$ and since L^0 does not intersect $|\dot{\lambda}|$ transversely, $\mathscr{V}(L^0) \cup \partial_{\chi=0}M$ does not contain any essential disc nor annulus. We have already shown that v_n is homotopic to the identity and that its restriction to $\partial_{\chi<0}M - \mathscr{V}(L^0)$ is isotopic to the inclusion map $\partial_{\chi<0}M - \mathscr{V}(L^0) \to M$. It follows from [Joh] that v_n is isotopic to the identity, concluding the proof that $b_{\mathscr{R}}$ is proper.

5 Consequences of Theorem 1.2

In this section, we will explain the proofs of Theorems 1.3 and 1.4.

Theorem 1.3. *Let $\mathring{\lambda} \in \mathscr{P}(M)/Mod_0(M)$ be a class of measured geodesic laminations such that $\mathscr{B}(\mathring{\lambda})$ is a single point ρ and let $\rho_n \subset AH(M)$ be a sequence of conjugacy classes representations. Then ρ_n converges strongly to ρ_∞ if and only if :*

- *there is N such that for $n \geq N$, ρ_n is a geometrically finite representation with bending measured geodesic lamination $\mathring{\lambda}_n \in \mathscr{P}(M)/Mod_0(M)$;*

- *the sequence $\ddot{\lambda}_n \in \mathscr{P}^+(M)/(Mod_0(M) * \mathscr{R})$ converges to $\ddot{\lambda}$.*

Proof. Let $\mathring{\lambda} \in \mathscr{P}(M)/Mod_0(M)$ be a class of measured geodesic laminations such that $\mathscr{B}(\lambda)$ is a single point ρ, and let $\lambda \in \mathscr{P}(M)$ be a representative of $\mathring{\lambda}$.

Let $\rho_n \subset AH(M)$ be a sequence of representations with the following properties : there is N such that for $n \geq N$, ρ_n is a geometrically finite representation with bending measured geodesic lamination $\mathring{\lambda}_n \in \mathscr{P}(M)/Mod_0(M)$ and the sequence $\ddot{\lambda}_n \in \mathscr{P}^+(M)/(Mod_0(M) * \mathscr{R})$ converges to $\ddot{\lambda}$. For $n \geq N$, let $\lambda_n \in \mathscr{P}(M)$ be a representative of $\mathring{\lambda}_n$ such that $\dot{\lambda}_n$ converges to $\dot{\lambda}$. For $n \geq N$, there is a geometrically finite metric σ_n with associated representation ρ_n and with bending measured geodesic laminations λ_n. By Theorem 1.2, any subsequence of σ_n contains a subsequence converging to a geometrically finite metric σ_∞. By [Le3], the bending measured geodesic lamination of σ_∞ is λ. Since $\mathscr{B}(\mathring{\lambda})$ is a single point ρ, the representation ρ is associated to σ_∞. By Lemma 4.11, the corresponding subsequence of ρ_n converges strongly (up to conjugacy) to ρ. We have thus proved that any subsequence of ρ_n contains a subsequence that converges strongly to ρ. It follows that ρ_n converges strongly to ρ.

Let $\rho_n \in AH(M)$ be a sequence of representation converging strongly to ρ. Let $\sigma \in \mathscr{GF}(M)$ be a geometrically finite representation whose bending measured geodesic lamination is λ. By [Ta], there is N such that for $n \geq N$, ρ_n is geometrically finite. Therefore there is a sequence of metrics $\sigma_n \in \mathscr{GF}(M)$ with associated representations ρ_n such that σ_n converges to σ in $\mathscr{GF}(M)$. Let λ_n be the bending measured geodesic lamination of σ_n, λ_n is a representative of the bending measured lamination $\mathring{\lambda}_n \in \mathscr{P}(M)/Mod_0(M)$ of ρ_n. By [Le3], the sequence $\dot{\lambda}_n$ converges to $\dot{\lambda}$. Therefore the sequence $\ddot{\lambda}_n \in \mathscr{P}^+(M)/(Mod_0(M) * \mathscr{R})$ converges to $\ddot{\lambda}$. □

Theorem 1.4. *The action of $Mod(M)$ on $\mathscr{D}(M)$ is properly discontinuous.*

Proof. If we assume the contrary, there are $\lambda \in \mathscr{D}(M)$, $(\lambda_n) \in \mathscr{D}(M)$ and $(\phi_n) \in Mod(M)$ such that (λ_n) and $(\phi_n(\lambda_n))$ converge simultaneously to λ in $\mathscr{ML}(\partial M)$ and that for any $n \neq m$, ϕ_n is not isotopic to ϕ_m.

First assume that there is no homeomorphism between M and an I-bundle over a pair of pants P such that $|\lambda|$ is mapped to a section of the bundle over ∂P. Since $\lambda \in \mathscr{D}(M)$, there is $\eta > 0$ such that $i(\lambda, \partial D) > \eta$ for any essential disc D. Let $\frac{2\pi}{\eta}\lambda$ be the measured geodesic lamination obtained by rescaling the measure of λ by $\frac{2\pi}{\eta}$. Let λ^i be a compact leaf of $\frac{2\pi}{\eta}\lambda$ with a weight greater than or equal to π. If, up to extracting a subsequence, λ^i is a compact leaf of all the measured geodesic laminations λ_n, replace, in $\frac{2\pi}{\eta}\lambda$ and in all $\frac{2\pi}{\eta}\lambda_n$, λ^i by a leaf with weight equal to π. Let λ'_∞ and λ'_n be the measured geodesic laminations obtained by doing this operation for all the leaves of $\frac{2\pi}{\eta}\lambda$ with a weight greater than or equal to π. We remark that λ'_∞ may have some leaves with a weight greater than π but that for n large enough, the compact leaves of λ'_n all have a weight less than or equal to π. Note also that (λ'_n) and $(\phi_n(\lambda'_n))$ converge simultaneously to λ'_∞ in $\mathscr{ML}(\partial M)$. By [Le2, Lemma 3.4], λ'_∞ and λ'_n satisfy conditions $b)$, $c)$. For n large enough, the λ'_n also satisfy condition $a)$ hence, by [Le1], there is a geometrically finite metric $\sigma_n \in \mathscr{GF}(M)$ whose bending measured lamination is (λ'_n). The bending measured geodesic lamination of $\phi_{n*}(\rho_n)$ is $\phi_n(\lambda'_n)$ and by construction $\phi_n(\lambda'_n) \longrightarrow \lambda'_\infty$. By Theorem 1.2, up to extracting a subsequence (σ_n) and $(\phi_{n*}(\sigma_n))$ converge in $\mathscr{GF}(M)$.

Therefore we can choose representations $\rho_n : \pi_1(M) \to Isom(\mathbb{H}^3)$ associated to σ_n such that ρ_n and $\phi_{n*} \circ \rho_n$ both converge algebraically. It is a classical result that the action of $Mod(M)/Mod_0(M)$ on the subset of $AH(M)$ made up by the geometrically finite representations is properly discontinuous. It follows that, up to extracting a subsequence, there is n_0 such that $\phi_{n_0}^{-1} \circ \phi_n$ is an element of $Mod_0(M)$.

Since (λ'_n) converges to λ'_∞, up to extracting a subsequence, the union $\lambda^{(p)}$ of the compact leaves of λ'_n with a weight equal to π does not depend on n. Since $(\phi_n(\lambda'_n))$ converges to λ'_∞, up to extracting a subsequence, $(\phi_n(\lambda^{(p)}))$ does not depend on n. Up to changing n_0 and extracting a subsequence, we may assume that $\phi_{n_0}^{-1} \circ \phi_n$ fixes $\lambda^{(p)}$ and that $\phi_{n_0}^{-1} \circ \phi_n$ maps each component of $\partial M - \lambda^{(p)}$ to itself.

Let S be a connected component of $\partial M - \lambda^{(p)}$. Let us denote by s_n^S the metric induced on $S \subset \partial M$ by a homeomorphism $h_n : M - (\lambda^{(p)} \cup \partial_{\chi=0} M) \to N(\sigma_n)$ associated to σ_n. We have seen in Lemma 4.3 that, up to extracting a subsequence, the sequence (s_n^S) converges algebraically. Since $\phi_{n_0}^{-1} \circ \phi_n(\lambda'_n)$ converges to $\phi_{n_0}^{-1}(\lambda'_\infty)$, it follows from the proof of Lemma 4.3 that the sequence $(\phi_{n_0}^{-1} \circ \phi_n)_* s_n^S$ converges algebraically. It is well known that the action of $Mod(S)$ on the isotopy classes of hyperbolic metrics on S is properly discontinuous. It follows that there is n_1 such that, up to extracting a subsequence, the restrictions of $\phi_{n_0}^{-1} \circ \phi_n$ and of $\phi_{n_0}^{-1} \circ \phi_{n_1}$ to $S \subset \partial M$ are isotopic.

All the homeomorphisms $\phi_{n_0}^{-1} \circ \phi_n$ are homotopic to the identity, hence they are homotopic to each other. In particular, $\phi_{n_0}^{-1} \circ \phi_n$ is homotopic to $\phi_{n_0}^{-1} \circ \phi_{n_1}$ for any n. We have seen in the previous paragraph that the restriction of $\phi_{n_0}^{-1} \circ \phi_n$ to each component of $\partial M - \lambda^{(p)}$ is isotopic to the restriction of $\phi_{n_0}^{-1} \circ \phi_{n_1}$. Let $\mathscr{V}(\lambda^{(p)})$ be an annular neighborhood of $\lambda^{(p)}$, since $\lambda \in \mathscr{D}(M)$, $(M, \mathscr{V}(\lambda^{(p)}))$ does not contain any annulus nor essential disc. It follows from [Joh, Proposition 27.1] that, up to extracting a subsequence and changing n_1, $\phi_{n_0}^{-1} \circ \phi_n$ is isotopic to $\phi_{n_0}^{-1} \circ \phi_{n_1}$ on the whole manifold M. This contradicts our hypothesis (that for any $n \neq m$, ϕ_n is not isotopic to ϕ_m) and concludes the proof in this case.

Finally assume that there is a homeomorphism between M and an I-bundle over a pair of pants P such that $|\lambda|$ is mapped to a section of the bundle over ∂P. Let $\eta > 0$ be such that $i(\partial D, \lambda) > \eta$ for any essential disc D. Let $\frac{2\pi}{\eta}\lambda$ be the measured geodesic lamination obtained by rescaling the measure of λ by $\frac{2\pi}{\eta}$. If a homoclinic leaf l intersects $|\lambda|$ only in one point then l contains two disjoint half leaves both spiraling in the same direction toward a leaf of λ. Such a leaf can not lie in a Hausdorff limit of simple closed curve. Furthermore $i(\partial E, \frac{2\pi}{\eta}\lambda) > 2\pi$ for any essential annulus E. It follows that the proof of Theorem 1.2 still works in this case yielding the following conclusion : if $\sigma_n \in \mathscr{GF}(M)$ is a sequence of metrics whose bending geodesic measured laminations λ_n converges to $\frac{2\pi}{\eta}\lambda$ then there is a subsequence such that σ_n converges to the fuchsian metric whose parabolics are the leaves of λ. Now we can apply the same arguments as in the other case. □

A Compactification of Teichmüller space and Λ-measured laminations

Let S be a hyperbolic surface with finite area and let (s_n) be a sequence of complete hyperbolic metrics on S. Assume that no subsequence of (s_n) converges algebraically. Then by [MoS1], there is a subsequence (which we will also denote by (s_n)) which tends to a small minimal action of $\pi_1(S)$ on a Λ-tree $\mathscr{T}$. The basic property of this action is that for any simple closed curves c_1, c_2 such that $l_{s_n}(c_1) \longrightarrow \infty$, we have $\lim_{n\longrightarrow\infty} \frac{l_{s_n}(c_2)}{l_{s_n}(c_1)} = \frac{\delta_{\mathscr{L}}(c_2)}{\delta_{\mathscr{L}}(c_1)}$. By [MoO], this action of $\pi_1(S)$ on $\mathscr{T}$ is dual to a Λ-measured geodesic lamination $\mathscr{L}$.

In this appendix, we will prove the continuity theorem for Λ-measured geodesic laminations which was used in the present paper.

Theorem A.1. *Let (s_n) be sequence of complete hyperbolic metrics of finite volume on a surface S tending to a Λ-measured lamination $\mathscr{L}$ and let (λ_n) be a sequence of measured geodesic laminations such that $(\dot\lambda_n)$ converges to some $\dot\lambda \in \mathscr{ML}(S)/\mathscr{R}$ and such that $l_{s_n}(\lambda_n)$ is bounded. Then $|\dot\lambda|$ does not intersect $|\mathscr{L}|$ transversely.*

Proof. Let (s_n), $\mathscr{L}$, (λ_n), λ be as in the statement and suppose that $|\lambda|$ intersects $|\mathscr{L}|$ transversely. Fix a reference hyperbolic metric on S and denote by λ^i a component of λ which intersects $\mathscr{L}$ transversely.

First assume that λ^i is an irrational lamination. Endow S with a complete hyperbolic metric and let $c \subset S(\lambda^i)$ be a simple closed geodesic. Since λ^i is arational in $S(\lambda^i)$, any leaf of $\mathscr{L}$ intersecting c transversely intersects λ^i transversely. Therefore, there is a train track τ with the following properties :

- τ carries λ^i;

- τ has only one switch $A \subset c - \mathscr{L}$;

- for any arc $k \subset \tau$ transverse to the ties such that $k \cap A = \partial k$, we have $\int_k d\mathscr{L} > 2\int_c d\mathscr{L}$.

Let $k \subset \lambda^i$ be an arc, parametrized by $[0,1]$, such that $k \cap A = \partial k$. Since $|\lambda^i|$ and c are uniquely defined for any hyperbolic metric on S and since $\partial k \subset |\lambda^i| \cap c$, this arc k is well-defined for any hyperbolic metric on S (see 1.1). Let us denote by b the branch of τ containing k and by $b(\lambda)$ the measure of a tie of b.

Assume that $\frac{dk}{dt}(0)$ and $\frac{dk}{dt}(1)$ point to the same side of A. Denote by κ the subsegment of A joining $k(0)$ to $k(1)$ and by c_b the simple closed geodesic homotopic to $k \cup \kappa$. Let $\hat d$ be a lift of $k \cup \kappa$ to $\mathbb{H}^2$, we may assume that A has been chosen small enough (with respect to the reference metric) so that $\hat d$ is a quasi-geodesic. Then there are a lift $\hat c_b$ of c_b and a number $\varepsilon > 0$ such that $\hat d$ lies in an ε-neighborhood of $\hat c_b$. Moreover $\hat c_b$ contains a family of segments $[a_i, a_{i+1}]$, $i \in \mathbb{Z}$ such that the ε-neighborhood of each $[a_i, a_{i+1}]$ contains one lift of k. Therefore any leaf of the preimage of $\mathscr{L}$ which intersects $\hat d$ intersects $\hat d$ only once and also intersects $\hat c_b$. Then we have $\delta_{\mathscr{L}}(c_b) = \delta_{\mathscr{L}}(k) > 2\delta_{\mathscr{L}}(c)$. Hence $l_{s_n}(c_b) - 2l_{s_n}(c)$ tends to ∞. So we have

$$l_{s_n}(k) \geq l_{s_n}(c_b) - l_{s_n}(A) \geq l_{s_n}(c_b) - l_{s_n}(c) \longrightarrow \infty. \tag{1}$$

This inequality (1) holds for any arc k of $\lambda^i - A$ lying in b. Up to extracting a subsequence, the laminations λ_n are carried by a train track containing τ as a subtrack and inequality (1) holds for any arc of $\lambda_n - A$ lying in b. So we have $l_{s_n}(\lambda_n) \geq b(\lambda_n)(l_{s_n}(c_b) - l_{s_n}(c))$. Since $b(\lambda_n)$ tends to $b(\lambda)$, for large n, we have $l_{s_n}(\lambda_n) \geq \frac{1}{2}b(\lambda)(l_{s_n}(c_b) - l_{s_n}(c)) \longrightarrow \infty$. This contradicts the assumption that $l_{s_n}(\lambda_n)$ is bounded and shows that $\frac{dk}{dt}(0)$ and $\frac{dk}{dt}(1)$ do not point to the same side of A.

Assume now that for any arc $k \subset \tau$ transverse to the ties such that $k \cap A = \partial k$, the tangent vectors $\frac{dk}{dt}(0)$ and $\frac{dk}{dt}(1)$ point to different sides of A. Let $k^1 \subset \tau$ and $k^2 \subset \tau$ be two arcs transverse to the ties such that we have $k^j \cap A = \partial k^j$ and $k^1 \cap k^2 = k^1(1) = k^2(0)$. Denote by b^j the branch of τ containing k^j. By our assumption, we have $b^1 \neq b^2$. Let $[k^1(0), k^2(1)] \subset A$ denote the segment joining $k^1(0)$ to $k^2(1)$. If $[k^1(0), k^1(1)] \cap (k^1 \cup k^2)$ is empty, let c_b be the simple closed geodesic isotopic to $k^1 \cup k^2 \cup [k^1(0), k^1(1)]$. Otherwise let c_b be the simple closed geodesic isotopic to $k^1 \cup [k^1(1), k^2(1)] \cup (k^2)^{-1} \cup [k^2(0), k^1(0)]$. By the same arguments as before we have $l_{s_n}(\lambda_n) \geq \frac{1}{2}(\int_{A \cap b^1 \cap b^2} d\lambda)(l_{s_n}(c_b) - 2l_{s_n}(c)) \longrightarrow \infty$. This yields the same contradiction as before implying that λ^i is not an irrational lamination.

Thus λ^i is a closed leaf. Since $l_{s_n}(\lambda^i)$ is bounded, λ^i is not a leaf of λ_n for large n. Endow S with a complete hyperbolic metric and consider two pairs of pants $P^1 \subset \partial M$ and $P^2 \subset \partial M$ such that $\lambda^i \subset int(P^1 \cup P^2)$. We will only deal with the case where P^1 and P^2 are distinct since only slight modifications are required to deal with the other case. Remove from P^j an annular neighborhood of $\partial P^j - \lambda^i$, for $j = 1, 2$, and denote by S the surface $P^1 \cup P^2$. We have $\partial S = \partial P^1 \cup \partial P^2 - \lambda^i$ and the points of $\partial S \cap |\mathscr{L}|$ are transverse intersections. Let $c \subset S$ be a simple closed geodesic which intersects λ^i in two points. Changing c by Dehn twists along λ^i if necessary, we may assume that $c \not\subset |\mathscr{L}|$. Let us denote by $|\mathscr{L}^i|$ the union of the components of $|\mathscr{L}| \cap S$ which intersect λ^i and by $|\mathscr{L}^r|$ the union of the remaining components of $|\mathscr{L}| \cap S$. The transverse Λ-measure of $\mathscr{L}$ induces transverse Λ-measures on $|\mathscr{L}^i|$ and on $|\mathscr{L}^r|$ and we have $\delta_{\mathscr{L}}(c) = \delta_{\mathscr{L}^i}(c) + \delta_{\mathscr{L}^r}(c)$. Since any component of $|\mathscr{L}| \cap P^j$ is compact, there is $m \in \mathbb{N}$ such that for $j = 1, 2$ and for any component l of $|\mathscr{L}| \cap P^j$, we have $\sharp\{l \cap c\} \leq \frac{m}{2}$. Therefore $\delta_{\mathscr{L}^i}(c) \leq m\delta\mathscr{L}(\lambda^i)$.

By [Th3], there is a sequence of measured geodesic laminations (ν_n) such that for any simple closed curve $d \subset S$, $\exists C_d$ such that $i(\nu_n, d) \leq l_{s_n}(d) \leq i(\nu_n, d) + C_d$. Let us divide $\nu_n \cap S$ into $\nu_n^i \cup \nu_n^r$ where ν_n^i is the union of the components which intersect λ^i. We have $i(\nu_n, c) = i(\nu_n^i, c) + i(\nu_n^r, c)$. Any connected component of $|\nu_n^r|$ intersects c once or twice, depending on whether its vertices lie in different components of ∂S or not. Hence there are $a_1, a_2, a_3, a_4 \in \{\frac{1}{2}\}\mathbb{N}$ and $\epsilon \in \{0, 1\}$ such that $i(\nu_n^r, c) = \sum_{k=1}^4 a_k i(\nu_n, \partial^k S) - \epsilon i(\nu_n, \lambda^i)$ where $\partial^k S$, $k = 1, 2, 3, 4$ are the connected components of ∂S. Let us explain how a_1 and a_2 are computed, considering that we have $\partial^1 S, \partial^2 S \subset \partial P^1$ (a_3 and a_4 are computed in the same way):

- if we have $i(\nu_n, \partial^1 S) + i(\nu_n, \partial^2 S) \geq i(\nu_n, \lambda^i)$, $i(\nu_n, \partial^2 S) + i(\nu_n, \lambda^i) \geq i(\nu_n, \partial^1 S)$ and $i(\nu_n, \partial^1 S) + i(\nu_n, \lambda^i) \geq i(\nu_n, \partial^2 S)$ then the total weight carried by the components of ν_n^r is $\frac{1}{2}(i(\nu_n, \partial^1 S) + i(\nu_n, \partial^2 S) - i(\nu_n, \lambda^i))$, therefore we set $a_1 = a_2 = \frac{1}{2}$ and $\epsilon = 1$ (see fig 5, case 1);

- if we have $i(\nu_n, \partial^1 S) + i(\nu_n, \partial^2 S) \leq i(\nu_n, \lambda^i)$ then we set $a_1 = a_2 = \epsilon = 0$ (see fig 5, case 2);

- if we have $i(\nu_n, \partial^2 S) + i(\nu_n, \lambda^i) \leq i(\nu_n, \partial^1 S)$ then we set $a_1 = 1$, $a_2 = 0$ and $\epsilon = 1$ (see fig 5, case 3). If we have $i(\nu_n, \partial^1 S) + i(\nu_n, \lambda^i) \leq i(\nu_n, \partial^2 S)$ then we set $a_1 = 0$, $a_2 = 1$ and $\epsilon = 1$.

If $i(\nu_n, \partial^1 S)$, $i(\nu_n, \partial^2 S)$ and $i(\nu_n, \lambda^i)$ satisfy an equality in the triangle inequalities, then we are in case 1 and in case 2 or in case 1 and in case 3 and we can choose the a_k and ϵ which are the most convenient for our purpose.

Up to extracting a subsequence, we may assume that the a_k and ϵ do not depend on n.

Let (κ_n) be a sequence of arcs of c such that $\underline{\lim} \int_{\kappa_n} d\lambda_n > 0$ and that (κ_n) converges for the Hausdorff topology to a point of λ^i. Let $\mathscr{V}(|\lambda^i|)$ be an annular neighborhood of $|\lambda^i|$ and recall the description of $\lambda_n \cap \mathscr{V}(|\lambda^i|)$ that we gave in the proof of Claim 3.8 : $|\lambda_n| \cap \mathscr{V}(|\lambda^i|)$ is a family of disjoint segments joining the two components of $\partial\mathscr{V}(|\lambda^i|)$ and turning a lot of times toward $|\lambda^i|$ (by Claim 2.2) and the total weight of the transverse measure carried by $|\lambda_n| \cap \mathscr{V}(|\lambda^i|)$ tends to 0 (by Claim 2.3). Given an orientation of c we order the points of $\kappa_n \cap \lambda_n$ by the induced order. For large n, κ_n contains an arc κ'_n such that we have $\int_{\kappa'_n} d\lambda_n \geq \frac{\int_{\kappa_n} d\lambda_n}{3}$ and that any point x_n of $\kappa'_n \cap \lambda_n$ is joined to a point $y_n > x_n$ of $\kappa_n \cap \lambda_n$ by an arc $]x_n, y_n[\subset \lambda_n - \kappa_n$. Fix n, consider a point x_n in $\kappa'_n \cap \lambda_n$ and the associated point y_n. Let $c_{x_n y_n}$ be the simple

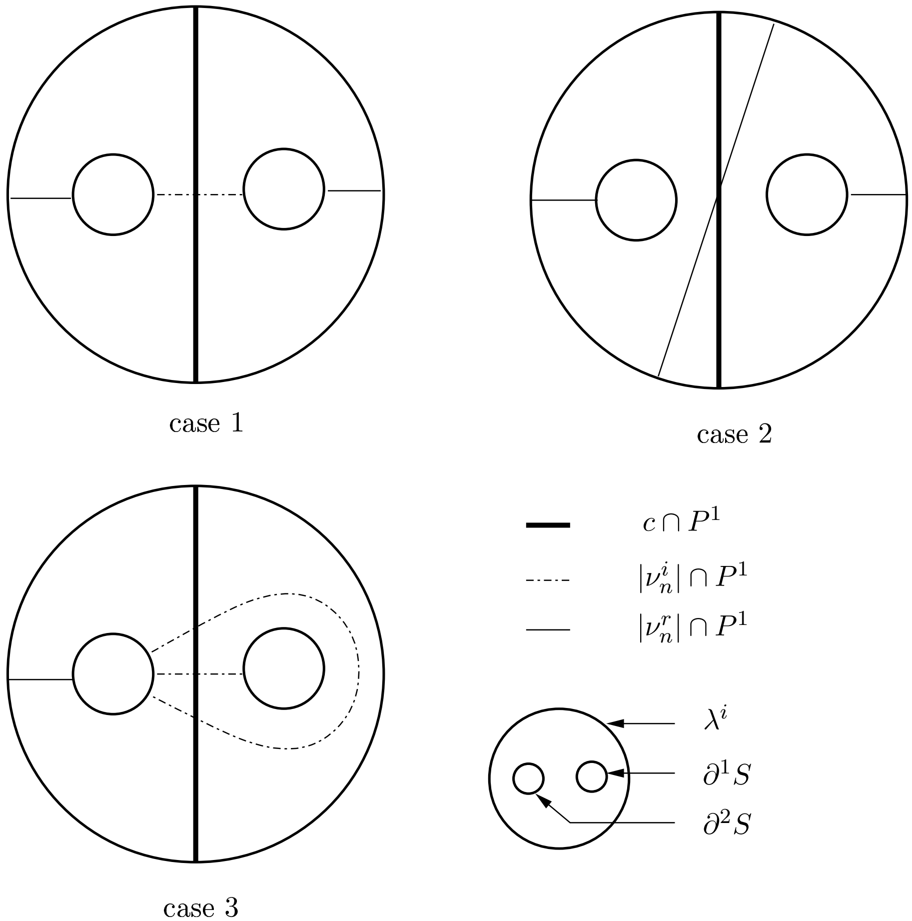

Figure 5: Computation of a_k and ϵ

closed curve which is the union of $[x_n, y_n] \subset \lambda_n$ and of the arc $\kappa_{x_n y_n} \subset \kappa_n$ joining x_n to y_n. This curve $c_{x_n y_n}$ can be homotoped in λ^i, hence we have $i(\nu_n, c_{x_n, y_n}) \geq i(\nu_n, \lambda^i)$. We have then $i(\nu_n, \lambda_n) \geq (\int_{\kappa'_n} d\lambda_n) \min\{\int_{[x_n, y_n]} d\nu_n / x_n \in \kappa'_n \cap \lambda_n\} \geq (\int_{\kappa'_n} d\lambda_n)(i(\nu_n, \lambda^i) - \int_{\kappa'_n} d\nu_n)$ hence $\int_{\kappa_n} d\nu_n \geq \int_{\kappa'_n} d\nu_n \geq i(\nu_n, \lambda^i) - \frac{3}{\int_{\kappa_n} d\lambda_n} i(\nu_n, \lambda_n)$. Let Q be the weight of λ^i as a sublamination of λ' and choose a number E with $0 < E < \frac{Q}{m+3}$. By the previous description of $\lambda_n \cap \mathscr{V}(|\lambda^i|)$, c contains $m+3$ disjoint segments κ_n^j such that $\int_{\kappa_n^j} d\lambda_n \geq E$ and such that for any $1 \leq j \leq m+3$, (κ_n^j) converges to a point $x \subset c \cap \lambda^i$ (which does not depend on j). It follows also from the description of $\lambda_n \cap \mathscr{V}$ that any leaf of $|\nu_n| \cap S$ intersecting a κ_n^j lies in $|\nu_n^i|$. Thus we have $i(\nu_n^i, c) \geq \sum_j \int_{\kappa_n^j} d\nu_n \geq (m+3) i(\nu_n, \lambda^i) - \frac{3(m+3)}{\int_{\kappa_n^j} d\lambda_n} i(\nu_n, \lambda_n)$.

Now we have
$i(\nu_n, c) \geq \sum_{k=1}^4 a_k i(\nu_n, \partial^k S) + (m+3-\epsilon) i(\nu_n, \lambda^i) - \frac{3(m+3)}{E} i(\nu_n, \lambda_n) \geq \sum_{k=1}^4 a_k i(\nu_n, \partial^k S) + (m+2-\epsilon) i(\nu_n, \lambda^i)$. This yields the inequalities : $l_{s_n}(c) \geq \sum_{k=1}^4 a_k (l_{s_n}(\partial^k S) - C_{\partial^k S}) + (m+2)(l_{s_n}(\lambda^i) - C_{\lambda^i}) \geq \sum_{k=1}^4 a_k l_{s_n}(\partial^k S) + (m+1) l_{s_n}(\lambda^i)$. By [MoS1], this implies the inequality :

$\delta_{\mathscr{L}}(c) \geq \sum_{k=1}^{4} a_k \delta_{\mathscr{L}}(\partial^k S) + (m+1)\delta_{\mathscr{L}}(\lambda^i)$.

As for the (ν_n) there exist $a_1, a_2, a_3, a_4 \in \{\frac{1}{2}\}\mathbb{N}$ and $\epsilon \in \{0,1,2\}$ such that we have $\delta_{\mathscr{L}^r} = \sum_{k=1}^{4} a_k \delta_{\mathscr{L}}(\partial^k S) - \epsilon \delta_{\mathscr{L}}(\lambda^i)$. If $i(\nu_n, \partial^j S)$ tens to ∞, then we have $\lim_{n \longrightarrow \infty} \frac{i(\nu_n, \partial^i S)}{i(,\nu_n, \partial^j S)} = \frac{\delta_{\mathscr{L}}(\partial^i S)}{\delta_{\mathscr{L}}(\partial^j S)}$. Therefore, for large n, $\delta_{\mathscr{L}}(\partial^k S)$ and $\delta_{\mathscr{L}}(\lambda^i)$ satisfy the same inequalities as $i(\nu_n, \partial^k S)$ and $i(\nu_n, \lambda^i)$. Hence the a_k are the same as before. Thus we have $\delta_{\mathscr{L}}(c) \leq \sum_{k=1}^{4} a_k \delta_{\mathscr{L}}(\partial^k S) + m\delta_{\mathscr{L}}(\lambda^i)$. This contradicts the inequality $\delta_{\mathscr{L}}(c) \geq \sum_{k=1}^{4} a_k \delta_{\mathscr{L}}(\partial^k S) + (m+1)\delta_{\mathscr{L}}(\lambda^i)$ and concludes the proof of Theorem A.1. □

Remark. An informal but more visual way to understand what is happening here is the following. The sequence of measured foliations ν_n tends to $\mathscr{L}$, namely for any simple closed curves c_1 and c_2 such that $i(\nu_n, c_1) \longrightarrow \infty$, we have $\lim_{n \longrightarrow \infty} \frac{i(\nu_n, c_2)}{i(,\nu_n, c_1)} = \frac{\delta_{\mathscr{L}}(c_2)}{\delta_{\mathscr{L}}(c_1)}$. The fact that $l_{s_n}(\lambda_n)$ is bounded forces λ_n to follow ν_n, which implies that λ does not intersect $\mathscr{L}$ transversely. ◇

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