

# QUASI-ISOMETRIC RIGIDITY OF THREE MANIFOLD GROUPS

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ABSTRACT. We provide a proof that the classes of finitely generated Kleinian groups and of three-manifold groups are quasi-isometrically rigid.

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## 1. INTRODUCTION

It was already known to Dehn that any finitely presented group can be realized as the fundamental group of a closed manifold of any dimension at least four, but this is not the case for three manifolds, e.g., the group  $\mathbb{Z}^4$  is not the fundamental group of any closed 3-manifold. This makes the class of three-dimensional manifolds special and we may expect that their fundamental groups enjoy specific properties which characterize them among finitely generated groups.

From the point of view of geometric group theory, one tries to understand the properties of a group by studying the different actions it admits on metric spaces. For the action of the group  $G$  on the geodesic metric space  $X$  to properly reflect the properties of  $G$ , we require that it is *geometric*: the group  $G$  acts by isometries (the action is distance-preserving), properly discontinuously (for any compact subsets  $K$  and  $L$  of  $X$ , at most finitely many elements  $g$  of  $G$  will satisfy  $g(K) \cap L \neq \emptyset$ ) and cocompactly (the orbit space  $X/G$  is compact). By identifying  $G$  with the orbit  $Go$  of a point  $o \in X$  and by pulling back the induced metric, we obtain a metric on  $G$ . Changing the orbit or the metric space  $X$  gives rise to new metrics. Thus we get a metric structure on  $G$  which is coarsely defined in the following sense.

A *quasi-isometry* between metric spaces  $X$  and  $Y$  is a coarsely bi-Lipschitz coarsely surjective map  $\varphi : X \rightarrow Y$ , i.e., there are constants  $\lambda > 1$  and  $c > 0$  such that:

- (quasi-isometric embedding) for all  $x, x' \in X$ , the two inequalities

$$\frac{1}{\lambda}d_X(x, x') - c \leq d_Y(\varphi(x), \varphi(x')) \leq \lambda d_X(x, y) + c$$

hold and

- (quasi-surjectivity) the  $c$ -neighborhood of the image  $f(X)$  covers  $Y$ .

This defines in fact an equivalence relation on (separable) metric spaces. The Švarc-Milnor lemma asserts that there is only one geometric action of a group on a proper geodesic metric space up to quasi-isometry [GdlH, Prop. 3.19]: we equip  $G$  with a reference metric induced by identifying  $G$  with an orbit  $Go$  under a geometric action on some proper geodesic space  $Y$  (usually one takes its left action on one of its locally finite Cayley graphs).

**Švarc-Milnor Lemma.** *Let  $X$  be a proper geodesic metric space. Let  $G$  act properly discontinuously and cocompactly on  $X$  by isometries. Then  $G$  is finitely generated and, for any  $x_0 \in X$ , the map  $g \mapsto g \cdot x_0$  is a quasi-isometry.*

Thus we have a coarsely well-defined metric on  $G$ , i.e., defined up to quasi-isometry, coming from its geometric actions. We are then naturally led to ask whether or not a property of a finitely generated group is invariant under quasi-isometries or, equivalently, whether or not a class of groups is *quasi-isometrically rigid*: a class of groups  $\mathcal{C}$  is quasi-isometrically rigid if any group quasi-isometric to a group in  $\mathcal{C}$  is in fact in  $\mathcal{C}$ .

Since the pioneering works of Stallings and Gromov, diverse classes of groups have been proved to be quasi-isometrically rigid. We should mention free groups [Sta1], nilpotent groups [Grv1], Abelian groups [Grv1, Bas, Gui, Pan, CTV] and word hyperbolic groups. A more thorough overview of these results will be given in §2.1.

In contrast with higher dimension (every finitely presented group is the fundamental group of a compact 4-manifold), fundamental groups of low-dimensional compact manifolds have many restrictive properties and their quasi-isometric rigidity is a challenging question that has already led to many interesting developments. In this paper we focus on compact 3-manifolds but the reader should be aware that the quasi-isometric rigidity of surface groups follows from [Gab], [CJ] and results we have already mentioned (see §2.1 for more details). Our main theorem completes the work of many people whose combined results can be fairly accurately summarized in the following statement; see §2.1 for more details about these results and §2.1 and §2.2 for the definition of irreducible 3-manifolds.

**Theorem 1.1.** *Let  $G$  be a group quasi-isometric to the fundamental group of a compact irreducible 3-manifold  $M$  with zero Euler characteristic. Then there is a short exact sequence*

$$1 \rightarrow F \rightarrow G \rightarrow Q \rightarrow 1$$

where  $F < G$  is a finite group and  $Q$  has a finite index subgroup isomorphic to the fundamental group of a compact 3-manifold with zero Euler characteristic.

As we will see in §4.2, it is relatively easy, using the work of [PW], to remove the assumption that  $M$  is irreducible. Thus extending Theorem 1.1 to manifolds with negative Euler characteristic completely settles the question. This is the purpose of the present paper which leads to the following statement:

**Theorem 1.2** (Quasi-isometric rigidity of 3-manifold groups). *The class of virtual fundamental groups of compact 3-manifolds is quasi-isometrically rigid. More precisely, a finitely generated group quasi-isometric to the fundamental group of a compact 3-manifold  $M$  contains a finite index subgroup isomorphic to the fundamental groups of a compact 3-manifold  $N$ .*

Notice that  $\pi_1(M)$  and  $\pi_1(N)$  need not be commensurable, i.e., may have no isomorphic finite index subgroups. For example, consider closed quotients of  $\mathbb{H}^2 \times \mathbb{E}^1$  and  $\widetilde{SL}_2(\mathbb{R})$ : since  $\mathbb{H}^2 \times \mathbb{E}^1$  and  $\widetilde{SL}_2(\mathbb{R})$  are quasi-isometric, all those quotients are quasi-isometric by Švarc-Milnor lemma, but  $\mathbb{H}^2 \times \mathbb{E}^1$  and  $\widetilde{SL}_2(\mathbb{R})$  have no isomorphic lattices, see [Wan, Lemma 6.3] and [Sco3, Theorem 5.2]. Non geometric examples can also be produced, using [Lee, KaL1, KaL2]. Nevertheless, it follows from the present work that, for a non-geometric irreducible manifold or a hyperbolic manifold with non-empty boundary, the fundamental groups of the pieces obtained after cutting the manifold along compressing discs, essential tori and annuli are well-defined up to commensurability.

Notice also that this statement involves a slight upgrade of Theorem 1.1 to go from a short exact sequence to a finite index subgroup. This comes from the following statement, interesting on its own right:

**Theorem 1.3.** *Let  $G$  be a finitely generated group and  $p : G \rightarrow Q$  a morphism with finite kernel. If  $Q$  has a finite index subgroup isomorphic to the fundamental group of a compact 2- or 3-manifold  $M$  then  $G$  is commensurable to  $Q$ .*

Two groups  $G$  and  $Q$  are *commensurable* if there are subgroups  $G' < G$  and  $Q' < Q$  of finite indices such that  $G'$  is isomorphic to  $Q'$ .

One of our main input deals with finitely generated Kleinian groups, i.e., discrete subgroups of  $\mathbb{P}SL_2(\mathbb{C})$ . We provide a new proof of the quasi-isometric rigidity of convex-cocompact Kleinian groups [Hai1, Hai2] which holds for all (finitely generated) Kleinian groups, leading to:

**Theorem 1.4** (Quasi-isometric rigidity of Kleinian groups). *The class of Kleinian groups is quasi-isometrically rigid. More precisely, a finitely generated group quasi-isometric to a Kleinian group contains a finite index subgroup isomorphic to a (possibly different) geometrically finite Kleinian group.*

To add a little perspective to this introduction let us remark that quasi-isometric groups may also be fairly different one from another. For instance, simplicity, linearity and residual finiteness [BM1], Kazhdan property ([Ger] and [DK, Theorem 19.76]) and Haagerup property [CAPV] are not virtually invariant under quasi-isometries. We refer the interested reader to [BM2], [dlH, §IV.50] and [DK, Chap. 25] for more properties that are known to be invariant or not under quasi-isometries.

When considering 3-manifold groups from the point of view of geometric group theory it is also natural to wonder about their classification up to quasi-isometry. This question has been the subject of extensive work of Behrstock and Neumann [BN1, BN2] who give a nearly complete answer.

**Outline of the paper.** The proofs Theorems 1.2 and 1.4 can be decomposed into three steps which are similar for both proofs. Let  $Q = \pi_1(M)$  be either the fundamental group of a compact 3-manifold (for Theorem 1.2) or a compact hyperbolic 3-manifold (for Theorem 1.4) and let  $G$  be a finitely generated group quasi-isometric to  $Q$ .

In the first step, we simultaneously split  $G$  and  $Q$  to get graphs of groups  $\mathcal{G} = (\Gamma_G, \{G_v\}, \{G_e\}, G_e \hookrightarrow G_{t(e)})$  and  $\mathcal{Q} = (\Gamma_Q, \{Q_v\}, \{Q_e\}, Q_e \hookrightarrow Q_{t(e)})$  (see §4.1 for definitions and notations) with quasi-isometric vertex groups. The splittings of  $Q$  come from topological splittings of  $M$  and the vertex groups  $Q_v$  are fundamental groups of hyperbolic 3-manifolds and of compact 3-manifolds with zero Euler characteristic in the proof of Theorem 1.2, while in the proof of Theorem 1.4, they are fundamental groups of pared acylindrical hyperbolic 3-manifolds and pared  $I$ -bundles.

The second step consists in showing that the classes to which the vertex groups  $Q_v$  belong are quasi-isometrically rigid. For Theorem 1.2, it is given by Theorems 1.4 and 1.1. For Theorem 1.4, it is the quasi-isometric rigidity of pared acylindrical Kleinian groups (Theorem 6.7) and pared  $I$ -bundles (Lemma 6.4). Applying these results to the vertex groups  $G_v$  we get finite index subgroups  $G'_v < G_v$  which are fundamental groups of compact 3-manifolds  $M_v$ .

In the third and last step we find a finite index subgroup  $G' < G$  whose intersection with each vertex group  $G_v$  is a finite index subgroup of  $G'_v$  and deduce that  $G'$  is the fundamental group of a compact 3-manifold obtained by gluing together finite covers of the manifolds  $M_v$ .

The arguments used in these steps are independent and for a better exposition we will study them in a different order. We will give more insights on these steps while we detail the plan of the paper.

First, in Section 2.1 we review results on quasi-isometric rigidity that are related to our topic. In particular, we recount the results that lead to Theorem 1.1. In Section 2.2 we provide background on three-manifolds and Kleinian groups. In particular, we show that a finitely generated Kleinian group is always isomorphic to a so-called geometrically finite Kleinian group with minimal parabolics, cf. Proposition 2.11. In Section 3 we introduce word hyperbolic and relative hyperbolic groups and establish some facts that will be used later on. In Section 4 we establish the first step described above, i.e., we study characteristic splittings of  $Q$  and  $G$  and their quasi-isometric invariance. In Section 4.2, we introduce a maximal splitting along spheres and discs and its analog for groups [Dun] which are used in the proofs of both Theorems 1.2 and 1.4. Its quasi-isometric invariance has been established in [PW]. In Section 4.3, we describe the characteristic torus decomposition and the induced splittings which are used in the proof of Theorem 1.2. The analog for groups and its quasi-isometric invariance are built on the work of Kapovich-Leeb [KaL3]. In Section 4.4, we introduce the characteristic annulus decomposition which is used in the proof of Theorem 1.4. Its analog for groups is defined using the work of [PS] on trees associated to the cut points and cut pairs of a continuum. The quasi-isometric invariance of the splitting thus produced is proved using the relation between the limit set of a Kleinian group and the characteristic submanifold of the underlying manifold [Wah], see also [Lec, §2.4]. In Section 5 we set up the third step by building finite index subgroups of fundamental groups of graph of groups with prescribed intersections with edge and vertex groups. The main tool used here is Wise's virtually special quotient theorem [Wis, Theorem 15.6], see also [Wis, Theorem 12.1]. In the last section, we conclude the proof of our main theorems. First we show Theorem 1.3 in Section 6.1 using an induction argument and hierarchies in groups quasi-isometric to 3-manifold groups. In Section 6.2, we establish the second step of the proof of Theorem 1.4 by proving the quasi-isometric rigidity of pared  $I$ -bundles and pared acylindrical Kleinian groups. Lastly, in Sections 6.3 and 6.4, we proceed with the proof of Theorems 1.4 and 1.2 as explained above.

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## 2. TOPOLOGY AND GEOMETRY OF 3-MANIFOLDS

**2.1. Quasi-isometric rigidity and geometric manifolds.** To complete our introduction, we will now give more details on the quasi-isometric rigidity results that were mentioned earlier.

The work of Stallings on ends of groups [Sta1] is a natural starting point. It leads to the quasi-isometric rigidity of virtually free groups, see [DK, Theorem 20.45]. A group  $G$  is said to *virtually* have a property if a finite index subgroup  $H$  of  $G$  has the said property.

**Theorem 2.1.** *The classes of virtually cyclic and virtually non-Abelian free groups are quasi-isometrically rigid.*

Another class of groups for which the quasi-isometric rigidity has been established is the class of virtually nilpotent groups. The next result follows from Gromov's polynomial growth theorem [Grv1]:

**Theorem 2.2** (Groups of polynomial growth). *The class of virtually nilpotent groups is quasi-isometrically rigid.*

Combining Theorem 2.2 with Bass-Guivarc'h formula for the polynomial growth of nilpotent groups [Bas, Gui] and the work of Pansu on their asymptotic cones [Pan], one gets [DK, Theorem 16.26]:

**Theorem 2.3.** *The class of virtually Abelian groups is quasi-isometrically rigid, with one quasi-isometry class for each rank.*

For a proof avoiding the classification of groups of polynomial growth, see [CTV].

2.1.1. *Surface groups.* Next, we explain the quasi-isometric rigidity of surface groups.

**Theorem 2.4.** *Let  $G$  be a group quasi-isometric to the fundamental group of a compact surface  $S$  (with or without boundary). Then there is a short exact sequence*

$$1 \rightarrow F \rightarrow G \rightarrow Q \rightarrow 1$$

*where  $F < G$  is a finite group and  $Q$  has a finite index subgroup isomorphic to the fundamental group of a compact surface.*

To make good use of the Švarc-Milnor lemma, we want to put a convenient metric on a compact manifold. In dimension 2, the Poincaré-Koebe uniformization theorem provides us with a spherical, Euclidean or hyperbolic metric for any compact surface. For better consistency with the 3 dimensional case, let us put that statement in the perspective of Thurston's model geometry.

A *model geometry*  $(G, X)$  is a manifold  $X$  together with a Lie group  $G$  of diffeomorphisms of  $X$ , such that:

- (a)  $X$  is connected and simply connected;
- (b)  $G$  acts transitively on  $X$ , with compact point stabilizers;
- (c)  $G$  is not contained in any larger group of diffeomorphisms of  $X$  with compact stabilizers of points; and
- (d) there exists at least one compact manifold modeled on  $(G, X)$ .

A *geometric structure* on a compact manifold  $M$  is a diffeomorphism from  $\text{int}(M)$  to  $X/\Gamma$  for some model geometry  $X$ , where  $\Gamma$  is a discrete subgroup of  $G$  acting freely on  $X$ . We say that a manifold is *geometric* if it has a geometric structure.

In dimension two there are three model geometries [Thu3, Theorem 3.8.2]:  $\mathbb{S}^2$ ,  $\mathbb{E}^2$  and  $\mathbb{H}^2$  together with their groups of isometries. Thus, the Poincaré-Koebe uniformization theorem says that all compact surfaces are geometric. Now the proof of Theorem 2.4 boils down to the rigidity of discrete subgroups of isometries of  $\mathbb{S}^2$ ,  $\mathbb{E}^2$  and  $\mathbb{H}^2$ .

**PROOF OF THEOREM 2.4.** As we have already explained,  $\pi_1(S)$  is isomorphic to a discrete subgroup  $\Gamma$  of  $Isom(X)$  with  $X = \mathbb{S}^2, \mathbb{E}^2$  or  $\mathbb{H}^2$ .

If  $X = \mathbb{S}^2$ ,  $G$  is finite and there is nothing to prove.

If  $X = \mathbb{E}^2$ ,  $\pi_1(S)$  is virtually Abelian of rank 1 or 2 depending on whether  $S$  has a boundary or not and the conclusion follows from Theorem 2.3; an alternate proof can be found in [Mes].

If  $X = \mathbb{H}^2$ ,  $\Gamma$  can be chosen to be a lattice of  $PSL_2(\mathbb{R})$  —a *Fuchsian group* of finite area. If  $\Gamma$  is non uniform (equivalently if  $\partial S \neq \emptyset$ ), it is a free group and the conclusion follows from Theorem 2.1. Thus we are only left with the case where  $G$  is quasi-isometric to a uniform lattice  $K < PSL_2(\mathbb{R})$ . Then  $K$  is word hyperbolic (see Definition 3.4) with boundary homeomorphic to  $S^1$ , so  $G$  as well. Therefore,  $G$  admits a uniform action on  $S^1$  and it follows from [CJ] or [Gab] that this action is conjugate to that of a cocompact Fuchsian group. Since  $G$  is a convergence group, the kernel of the action is finite. ■

Combining Theorems 2.4 and 1.3, we get:

**Theorem 2.5.** *Let  $G$  be a group quasi-isometric to the fundamental group of a compact surface  $S$  (with or without boundary). Then  $G$  has a finite index subgroup isomorphic to the fundamental group of a compact surface.*

2.1.2. *Three-manifold groups.* For fundamental groups of 3-manifolds, we summarized in the introduction the state of the art with the following statement:

**Theorem 1.1.** *Let  $G$  be a group quasi-isometric to the fundamental group of a compact irreducible 3-manifold  $M$  with zero Euler characteristic. Then there is a short exact sequence*

$$1 \rightarrow F \rightarrow G \rightarrow Q \rightarrow 1$$

where  $F < G$  is a finite group and  $Q$  has a finite index subgroup isomorphic to the fundamental group of a compact 3-manifold with zero Euler characteristic.

A 3-manifold is *irreducible* if every embedded sphere bounds a ball (the rest of the terminology used below is given in Section 2.2). As in the surface case, a first step in the proof consists in equipping 3-manifolds with convenient metrics.

Thurston has shown [Thu3, Theorem 3.8.4] that there are eight three-dimensional model geometries  $(G, X)$  which are  $\mathbb{S}^3, \mathbb{E}^3, \mathbb{H}^3, \mathbb{S}^2 \times \mathbb{E}^1, \mathbb{H}^2 \times \mathbb{E}^1, Nil, \widetilde{SL}(2, \mathbb{R}), Sol$  together with their groups of isometries.

In general, 3-manifolds are not geometric but the geometrization theorem (proved for Haken manifolds and stated as a conjecture in general by Thurston [Thu2] and proved by Perelman in general [BBM<sup>+</sup>, KIL, MT]) asserts that they can be decomposed into geometric pieces.

**Theorem 2.6** (Geometrization). *Every oriented irreducible compact 3-manifold can be cut along tori, so that the interior of each of the resulting manifolds has a geometric structure.*

A *geometric decomposition* of a 3-manifold  $M$  is a collection of essential tori  $T$  such that each component of  $M \setminus T$  has a geometric structure. The geometrization theorem precisely states that such a geometric decomposition always exists. We say that a geometric decomposition  $T$  is *minimal* if for any component  $T_1$  of  $T$ ,  $T \setminus T_1$  is not a geometric decomposition.

Let us now explain the main steps of the proof of Theorem 1.1, starting with geometric 3-manifolds.

**Theorem 2.7.** *Let  $G$  be a group quasi-isometric to the fundamental group of a geometric 3-manifold  $M$  with non-negative Euler characteristic. Then there is a short exact sequence*

$$1 \rightarrow F \rightarrow G \rightarrow Q \rightarrow 1$$

where  $F < G$  is a finite group and  $Q$  has a finite index subgroup isomorphic to the fundamental group of a geometric 3-manifold with zero Euler characteristic.

PROOF. We have already seen that  $M$  is modeled on one of the homogeneous spaces  $X = \mathbb{S}^3, \mathbb{S}^2 \times \mathbb{E}^1, \mathbb{E}^3, \mathbb{H}^2 \times \mathbb{E}^1, \mathbb{H}^3, Nil, Sol, \widetilde{SL_2(\mathbb{R})}$ .

When  $X = \mathbb{S}^3$ , then  $\pi_1(M)$  is finite.

When  $X = \mathbb{S}^2 \times \mathbb{E}^1$ , then  $\pi_1(M)$  is virtually cyclic and the conclusion follows from Theorem 2.1.

When  $X = \mathbb{E}^3$ , it is a special case of Theorem 2.3.

When  $X = Nil$ , it follows from Theorem 2.2 and results of Mal'cev, Guivarc'h [Gui] and Jenkins [Jen], cf. [Fri, Theorem 1.7].

When  $X = \mathbb{H}^3$ , it is due to Sullivan [Sul] and Cannon-Cooper [CC] when  $\partial M = \emptyset$  and Schwartz [Sch] when  $\partial M \neq \emptyset$ .

When  $X = Sol$ , it has been proved by Eskin, Fisher and Whyte [EFW].

Finally, when  $X = \mathbb{H}^2 \times \mathbb{E}^1$  and  $X = \widetilde{SL_2(\mathbb{R})}$ , it is due to Rieffel [Rie], see also [KaL3, §5.2]. Note that these two geometries are quasi-isometric. ■

Notice that geometric 3-manifolds with negative Euler characteristic are hyperbolic, i.e., modeled on  $\mathbb{H}^3$ .

From a strong invariance of the geometric decomposition under quasi-isometry, Kapovich-Leeb [KaL3] deduce the quasi-isometric rigidity of fundamental groups of irreducible non-geometric 3-manifolds with zero Euler characteristic. We will give more details in §4.3 since we will use this invariance to prove Theorem 1.2.

**Theorem 2.8** (Kapovich-Leeb). *Let  $G$  be a group quasi-isometric to the fundamental group of an irreducible non-geometric Haken compact 3-manifold  $M$  with zero Euler characteristic. Then there is a short exact sequence*

$$1 \rightarrow F \rightarrow G \rightarrow Q \rightarrow 1$$

where  $F < G$  is a finite group and  $Q$  has a finite index subgroup isomorphic to the fundamental group of an irreducible non-geometric compact 3-manifold  $M$  with zero Euler characteristic.



By the geometrization theorem (Theorem 2.6), any non-geometric compact 3 manifold is Haken (see the next section for a definition). Thus we get Theorem 1.1 simply by combining Theorems 2.7 and 2.8.

**2.2. Three-manifolds and groups.** In this section and the next one, we will review some classical definitions and results that are used in this paper. We start with 3-manifold topology, basic references include [Jac, Hem1].

*2.2.1. Three-manifold topology.* Let  $M$  be a compact orientable 3-manifold with boundary or not. An embedded surface  $(S, \partial S) \rightarrow (M, \partial M)$  is *incompressible* if the inclusion  $i : S \rightarrow M$  gives rise to an injective morphism  $i_* : \pi_1(S, x) \rightarrow \pi_1(M, x)$ . The *double* of a manifold  $M$  with boundary is the union of  $M$  and of a copy of itself glued along its boundary. An embedded surface  $S$  in  $M$  is *boundary incompressible* if its double is incompressible in the double of  $M$ . A surface  $S$  in  $M$  is *non-peripheral* if it is *properly embedded*, i.e.  $S \cap \partial M = \partial S$ , and if the inclusion  $i : S \rightarrow M$  is not homotopic relatively to  $\partial S$  to a map  $f : S \rightarrow M$  such that  $f(S) \subset \partial M$ . A surface  $S$  is *essential* if it is properly embedded, two-sided, incompressible, boundary incompressible, non-peripheral and does not bound a 3-ball. An essential disc is also called a *compression disc*.

The manifold  $M$  is *irreducible* if it contains no essential sphere, equivalently every embedded sphere bounds a ball. We say that  $M$  is *boundary irreducible* or  *$\partial$ -irreducible* if each component of  $\partial M$  is incompressible. By results of Kneser and Stallings,  $M$  is irreducible and boundary irreducible if and only if  $\pi_1(M)$  is one-ended.

The manifold  $M$  is *Haken* if it contains an essential surface. If  $\partial M \neq \emptyset$  then  $M$  is necessarily Haken, see [Hem1, Chap. 13].

A 3-manifold  $M$  is *atoroidal* if every subgroup of  $\pi_1(M)$  isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}$  is conjugate to a subgroup of the fundamental group of a boundary component. As we will see in the next section this property characterizes hyperbolic manifolds. An *acylindrical compact manifold* is atoroidal has incompressible boundary and no essential annuli.

A *Seifert manifold* is a compact 3-manifold which admits a foliation by circles. Even though this is not the classical definition, it is equivalent to it [Eps]. A *graph manifold* is a compact irreducible  $\partial$ -irreducible 3-manifold with no atoroidal pieces in its torus decomposition, see §4.3.

A compact *pared manifold*  $(M, P)$  is given by a compact irreducible 3-manifold  $M$  together with a paring  $P \subset \partial M$  which is a finite collection of pairwise disjoint incompressible annuli and tori satisfying:

- every Abelian, non cyclic subgroup of  $\pi_1(M)$  is conjugate to a subgroup of the fundamental group of a component of  $P$  and
- any incompressible cylinder  $(C, \partial C) \subset (M, P)$  can be homotoped relatively to its boundary into  $P$ .

Notice that by definition only atoroidal manifolds can have a paring. We say that  $(M, P)$  is *acylindrical* if there is no essential disc or cylinder in  $M$  disjoint from  $P$ .

**2.2.2. Kleinian groups.** A *Kleinian group*  $K$  is a discrete subgroup of  $\mathbb{P}SL_2(\mathbb{C})$  —the group of orientation preserving isometries of the 3-dimensional hyperbolic space  $\mathbb{H}^3$ . An orientable compact 3-manifold  $M$  is *hyperbolizable* if its interior is homeomorphic to the quotient  $\mathbb{H}^3/K$  where  $K$  is a torsion free Kleinian group. Such a manifold  $M$  is irreducible and atoroidal and we say that  $M$  is *uniformized* by  $K$ . Note that  $K$  is isomorphic to the fundamental group of  $M$ , and that it is necessarily word hyperbolic if it contains no subgroup isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}$ , see Section 3.2. On the other hand, the tameness theorem [Ago1, CG] asserts that, when  $K$  is a finitely generated torsion free Kleinian group,  $\mathbb{H}^3/K$  is homeomorphic to the interior of a compact 3-manifold  $M_K$  (with fundamental group isomorphic to  $K$ ) that we call the *Kleinian manifold of  $K$* .

As Poincaré observed, we may identify the Riemann sphere with the boundary at infinity of  $\mathbb{H}^3$  [Poi]. Then  $K$  acts on the Riemann sphere  $\widehat{\mathbb{C}}$  via Möbius transformations. The latter action is usually not properly discontinuous: there is a canonical and invariant partition

$$\widehat{\mathbb{C}} = \Omega_K \sqcup \Lambda_K$$

where  $\Omega_K$  denotes the *ordinary set*, which is the largest open set of  $\widehat{\mathbb{C}}$  on which  $K$  acts properly discontinuously, and where  $\Lambda_K$  denotes the *limit set*, which is the minimal  $K$ -invariant compact subset of  $\widehat{\mathbb{C}}$  (when  $K$  is infinite). The construction of the Kleinian manifold  $M_K$  induces an embedding  $(\mathbb{H}^3 \cup \Omega_K)/K \hookrightarrow M_K$  whose image is the complement of a subsurface of  $\partial M_K$ .

The group  $K$  preserves the convex hull  $\text{Hull}(\Lambda_K)$  of its limit set in  $\mathbb{H}^3$ . The group  $K$  is *convex-cocompact* if its action is cocompact on  $\text{Hull}(\Lambda_K)$  and  $K$  is *geometrically finite* if a regular neighborhood of its convex core  $\text{Hull}(\Lambda_K)/K$  has finite volume. Ahlfors [Ahl] showed that if the limit set of a geometrically finite Kleinian group is not the whole Riemann sphere then it has measure 0 (this holds more generally for finitely generated Kleinian groups by [Can], [Ago1] and [CG]). When  $K$  is not Fuchsian, i.e.,  $\text{Hull}(\Lambda_K)$  is not contained in a geodesic plane, there is a homeomorphism between  $(\mathbb{H}^3 \cup \Omega_K)/K$  and the *convex core*  $\text{Hull}(\Lambda_K)/K$  constructed using the closest point projection. As we have explained above  $(\mathbb{H}^3 \cup \Omega_K)/K$  embeds in  $M_K$ , hence the convex core also embeds in the Kleinian manifold. When  $K$  is geometrically finite, the image of this embedding is the complement of a paring  $P$  of  $M_K$  corresponding to the parabolics of  $K$  (notice that this gives an alternate characterization of geometric finiteness). We say that a geometrically finite Kleinian group  $K$  *uniformizes* the pared manifold  $(M, P)$  when  $(\mathbb{H}^3 \cup \Omega_K)/K$  is homeomorphic to  $M \setminus P$ .

Thurston’s uniformization theorem (extended to non Haken manifolds by Perel’man) gives a topological characterization of hyperbolic manifolds. We will use the following form, see [Mor, Theorem B’], and also [Ota2, Ota3, Kap]:

**Theorem 2.9.** *Let  $(M, P)$  be a Haken pared 3-manifold. There is a geometrically finite, complete hyperbolic manifold  $N$  whose convex core is homeomorphic to  $M \setminus P$ .*

When  $(M, P)$  is acylindrical, then a doubling argument shows that we can require the convex core to have totally geodesic boundary [Thu1, Thm 3].

**Theorem 2.10.** *Let  $(M, P)$  be a Haken acylindrical pared 3-manifold. There is a geometrically finite, complete hyperbolic manifold  $N$  whose convex core is homeomorphic to  $M \setminus P$  and has totally geodesic boundary.*

We say that a Kleinian group is *minimally parabolic* if every parabolic subgroup is a rank 2 Abelian subgroup. Combining Theorem 2.9 with Scott’s core theorem we get that any finitely generated Kleinian group has a minimally parabolic version:

**Proposition 2.11.** *Any finitely generated Kleinian group is isomorphic to a geometrically finite, minimally parabolic, Kleinian group.*

PROOF. Let  $K$  be a finitely generated Kleinian group. By Scott’s core theorem [Sco1], the hyperbolic manifold  $\mathbb{H}^3/K$  contains a compact submanifold  $C$  such that the inclusion is a homotopy equivalence. If  $\partial C = \emptyset$ , then  $C = \mathbb{H}^3/K$  and  $\mathbb{H}^3/K$  is compact. Thus,  $K$  has no parabolic subgroup and there is nothing to prove. So let us assume that  $\partial C \neq \emptyset$ , in particular,  $C$  is Haken. A maximal Abelian non cyclic subgroup of  $\pi_1(C)$  corresponds to a rank 2 parabolic subgroup of  $K$ . By [McC, Theorem 2] we can change  $C$  by a homotopy so that such a subgroup is the fundamental group of a component of  $\partial C$ . It follows that  $C$  is atoroidal. Now if  $P$  is the union of the tori in  $\partial C$ , then  $(C, P)$  is a pared manifold and we conclude with Theorem 2.9. ■

### 3. HYPERBOLICITY IN THE SENSE OF GROMOV

Hyperbolic spaces and groups were introduced by Gromov in [Grv2] and have known many developments since. In this section, we will introduce hyperbolic spaces, hyperbolic and relative hyperbolic groups and a few more objects associated to these groups. We will also establish various facts that will be used throughout the paper. We start with hyperbolic spaces and word hyperbolic groups. Background on those includes [Grv2, GdlH, KB].

**3.1. Hyperbolic spaces.** Let  $X$  be a metric space. It is *geodesic* if any pair of points  $\{x, y\}$  can be joined by a (geodesic) segment i.e, a map  $\gamma : [0, d(x, y)] \rightarrow X$  such that  $\gamma(0) = x$ ,  $\gamma(d(x, y)) = y$  and  $d(\gamma(s), \gamma(t)) = |t - s|$  for all  $s, t \in [0, d(x, y)]$ . The metric space  $X$  is *proper* if closed balls of finite radius are compact.

A *triangle*  $\Delta$  in a metric space  $X$  is given by three points  $\{x, y, z\}$  and three (geodesic) segments (or sides) joining them two by two. Given a constant  $\delta \geq 0$ , the triangle  $\Delta$  is  $\delta$ -*thin* if any side of the triangle is contained in the  $\delta$ -neighborhood of the two others.

**Definition 3.1** (Gromov hyperbolic spaces). *A geodesic metric space is (Gromov) hyperbolic if there exists  $\delta \geq 0$  such that every triangle is  $\delta$ -thin.*

Basic examples of hyperbolic spaces are the complete simply connected hyperbolic manifolds  $\mathbb{H}^n$ ,  $\mathbb{R}$ -trees and their convex subsets. It follows from the shadowing lemma (see below) that, among geodesic metric spaces, hyperbolicity is invariant under quasi-isometry : if  $X, Y$  are two quasi-isometric geodesic metric spaces, then  $X$  is hyperbolic if and only if  $Y$  is hyperbolic. A  $(\lambda, c)$ -*quasigeodesic* is the image of an interval by a  $(\lambda, c)$ -quasi-isometric embedding.

**Lemma 3.2** (Shadowing lemma). *Given  $\delta, \lambda$  and  $c$ , there is a constant  $H = H(\delta, \lambda, c)$  such that, for any  $(\lambda, c)$ -quasigeodesic  $q$  in a proper geodesic  $\delta$ -hyperbolic metric space  $X$ , there is a geodesic  $\gamma$  at Hausdorff distance at most  $H$  from  $q$ .*

3.1.1. *Boundaries of hyperbolic spaces.* A proper (geodesic) hyperbolic space  $X$  admits a canonical compactification  $X \sqcup \partial X$  at infinity. This compactification can be defined by looking at the set of geodesic rays, i.e., isometric embeddings  $r : \mathbb{R}_+ \rightarrow X$ , up to bounded Hausdorff distance. The topology is induced by the uniform convergence on compact subsets. The boundary can be endowed with a family of *visual distances*  $d_v$  compatible with its topology, i.e., which satisfy

$$d_v(a, b) \asymp e^{-\varepsilon d(w, (a, b))}$$

where  $w \in X$  is any choice of a base point,  $\varepsilon > 0$  is a *visual parameter* chosen small enough, and  $(a, b)$  is any geodesic asymptotic to rays defining  $a$  and  $b$ .

If  $\Phi : X \rightarrow Y$  is a quasi-isometry between two proper hyperbolic spaces, then the shadowing lemma implies that  $\Phi$  induces a homeomorphism  $\phi : \partial X \rightarrow \partial Y$ . Actually, the trace map at infinity of a quasi-isometry is *quasi-Möbius* [Väi], i.e., there exists a homeomorphism  $\theta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that, for any distinct points  $x_1, x_2, x_3, x_4 \in \partial X$ ,

$$[\phi(x_1) : \phi(x_2) : \phi(x_3) : \phi(x_4)] \leq \theta([x_1 : x_2 : x_3 : x_4])$$

where

$$[x_1 : x_2 : x_3 : x_4] = \frac{|x_1 - x_2| \cdot |x_3 - x_4|}{|x_1 - x_3| \cdot |x_2 - x_4|}.$$

Quasi-Möbius maps are stable under composition.

Quasi-isometries provide natural examples of quasi-Möbius maps, cf. [Pau, Prop. 4.6]:

**Theorem 3.3** (Efremovic-Tihomirova, [ET]). *A  $(\lambda, c)$ -quasi-isometry between proper hyperbolic spaces extends as a  $\theta$ -quasi-Möbius map between their boundaries, where  $\theta$  only depends on  $\lambda, c$ , the hyperbolicity constants and the visual parameters.*

3.1.2. *Groups of isometries.* Let  $G$  be a group of isometries of a proper hyperbolic space  $X$ . It follows from Theorem 3.3 that the action of  $G$  extends to an action on  $\partial X$  by uniform quasi-Möbius mappings.

If the action of  $G$  is furthermore properly discontinuous on  $X$ , then the action of  $G$  on  $\partial X$  is a *convergence action* i.e., its diagonal action on the set of distinct triples is properly discontinuous [Fre, Tuk]. As for Kleinian groups, the *limit set*  $\Lambda_G$  is by definition the unique minimal closed invariant subset of  $X$  when  $G$  is infinite. It is empty if  $G$  is finite. One may also consult [Bow5] for basic properties of convergence groups.

3.2. **Word hyperbolic groups.** A properly discontinuous group action by isometries on a proper geodesic metric space is *geometric* if it is *cocompact*, i.e., if the quotient is compact.

**Definition 3.4** (Word hyperbolic groups). *A group  $G$  is word hyperbolic, or just hyperbolic for simplicity, if it admits a geometric action on a proper geodesic hyperbolic metric space.*

We note that since the hyperbolicity of a proper geodesic hyperbolic metric space is a quasi-isometric invariant property, the hyperbolicity of a group does not depend on the space it is acting upon. In particular, by the Švarc-Milnor lemma,  $G$  is finitely generated and its hyperbolicity can also be read from any of its locally finite Cayley graphs. Moreover, this implies the quasi-isometric rigidity of the class of word hyperbolic groups.

Fundamental groups of closed hyperbolic manifolds are obvious examples. Convex cocompact Kleinian groups are also hyperbolic by definition: a convex-compact Kleinian group  $G$  has a properly discontinuous and cocompact action on the convex hull  $\text{Hull } \Lambda_G$  of its limit set which is a hyperbolic space in the sense of Gromov.

The definition and the Švarc-Milnor lemma also imply that a word hyperbolic group  $G$  admits a topological boundary  $\partial G$  defined by considering the boundary of any proper geodesic metric space on which  $G$  acts geometrically. In the case of a convex-cocompact Kleinian group  $K$ , a model for the boundary  $\partial K$  is given by its limit set  $\Lambda_K$ .

The action of a hyperbolic group on its boundary is a *uniform convergence action*, i.e., its diagonal action on the set of distinct triples is not only properly discontinuous but also cocompact, cf. [Bow5].

These properties characterize word hyperbolic groups and their boundaries:

**Theorem 3.5** (Bowditch [Bow3]). *Let  $G$  be a convergence group acting on a perfect metrizable space  $X$ . The action of  $G$  is uniform on its limit set  $\Lambda_G$  if and only if  $G$  is word hyperbolic and if, furthermore, there exists an equivariant homeomorphism between  $\Lambda_G$  and the boundary at infinity  $\partial G$  of  $G$ .*

A general principle asserts that a word hyperbolic group is determined by its boundary. More precisely, Paulin proved that the quasi-isometry class of a word hyperbolic group is determined by its boundary equipped with its quasiconformal structure [Pau].

**Theorem 3.6** (Paulin [Pau]). *Two non-elementary word hyperbolic groups are quasi-isometric if and only if there is a quasi-Möbius homeomorphism between their boundaries.*

**3.3. Relative hyperbolicity.** The idea behind relatively hyperbolic groups which will be defined next is that some metric spaces such as Cayley graphs are hyperbolic away from some “codimension 1” subspaces or subgroups. Background on relatively hyperbolic groups includes [Grv2, Bow6, Hru] and the references therein. Let us first recall the definition of the *Busemann function*  $\beta_p(x, y)$  centered at a point  $p \in \partial X$  at infinity of a hyperbolic space  $X$  measured at two points  $x, y \in X$ :

$$\beta_p(x, y) = \sup \left\{ \lim_{t \rightarrow \infty} [|x - \gamma(t)| - t] : \gamma \text{ geodesic ray asymptotic to } p \text{ such that } \gamma(0) = y \right\}.$$

Given  $p \in \partial X$ , a base point  $w \in X$  and  $r \in \mathbb{R}$ , the *horoball centered at  $p$  of level  $r$*  is defined as

$$H(p, r) = \{x \in X : \beta_p(w, x) \leq r\}.$$

Let  $G$  be a group and let  $\mathbb{P}$  be a collection of subgroups. The pair  $(G, \mathbb{P})$  is *relatively hyperbolic* if there exists a hyperbolic proper geodesic metric space  $X$  on which  $G$  acts properly discontinuously by isometries and if there is a collection  $\mathcal{H}$  of  $G$ -invariant pairwise disjoint horoballs with the following properties:

- (1) any  $P \in \mathbb{P}$  is the stabilizer of the center (at infinity) of a horoball in  $\mathcal{H}$ ;
- (2) the stabilizer of any center of a horoball of  $\mathcal{H}$  is conjugate to a subgroup from  $\mathbb{P}$ ;
- (3) the action of  $G$  on  $X \setminus Y$  is cocompact, where we let  $Y$  denote the union of the horoballs in  $\mathcal{H}$ .

Such an action of  $G$  is called *cuspid uniform*. Note that we may assume that no two elements of  $\mathbb{P}$  are conjugate. The subgroups in  $\mathbb{P}$  are called *peripheral subgroups*.

**3.3.1. Horoballing, cusped spaces and Bowditch boundaries.** In this section, we introduce an explicit cusp uniform action for a relatively hyperbolic group. We first concentrate on the construction of horoballs resting on peripheral subgroups. Let  $(P, d)$  be a graph endowed with the path metric that makes each edge isometric to  $[0, 1]$ . Define its *horoballing space*  $H_P$  following Groves and Manning [GM1] as the graph modelled on  $P \times \mathbb{N}$  with additional edges

$$\begin{pmatrix} x \\ m \end{pmatrix} \sim \begin{pmatrix} y \\ m \end{pmatrix} \quad \text{if } d(x, y) \leq 2^m$$

and

$$\begin{pmatrix} x \\ m \end{pmatrix} \sim \begin{pmatrix} x \\ m+1 \end{pmatrix}$$

This new graph  $H_P$  is also endowed with the path metric so that each edge is isometric to  $[0, 1]$ . By [GM1, Theorem 3.8],  $H_P$  is hyperbolic.

Let  $G$  be a finitely generated group, let  $\mathbb{P} = \{P_1, \dots, P_n\}$  be a (finite) family of finitely generated subgroups of  $G$ , and let  $S$  be a finite generating set for  $G$  so that  $P_i \cap S$  generates  $P_i$  for each  $i \in \{1, \dots, n\}$ , and denote by  $\text{Cay}(G, S)$  the Cayley graph of  $(G, S)$ . For each  $i \in \{1, \dots, n\}$ , let  $T_i$  be a left transversal for  $P_i$ , i.e., a collection of representatives for left cosets of  $P_i \in G$  which contains exactly one element of each left coset. For each  $i$ , and each  $t \in T_i$ , let  $\text{Cay}_{i,t}$  be the full subgraph of the Cayley graph  $\text{Cay}(G, S)$  which contains  $tP_i$ . Each  $\text{Cay}_{i,t}$  is isomorphic to the Cayley graph of  $P_i$  with respect to the generators  $P_i \cap S$ . We define the *cusped space*

$$\text{Cus}(G, \mathbb{P}, S) = \text{Cay}(G, S) \cup (\{\cup H_{\text{Cay}_{i,t}} \mid 1 \leq i \leq n, t \in T_i\}),$$

where the graphs  $\text{Cay}_{i,t} \subset \text{Cay}(G, S)$  and  $\text{Cay}_{i,t} = \text{Cay}_{i,t} \times \{0\} \subset H_{\text{Cay}_{i,t}}$  are identified in the obvious way. When the choice of the generating set does not matter, we simplify the notations to  $\text{Cay}(G)$  and  $\text{Cus}(G, \mathbb{P})$ .

The cusped space provides a canonical way to construct a hyperbolic space on which a relatively hyperbolic group has a cusp uniform action. We follow Groves and Manning [GM1] in using this space to give a different characterization of relatively hyperbolic groups.

**Theorem 3.7** (Groves and Manning [GM1]). *Let  $G$  be a finitely generated group, let  $\mathbb{P} = \{P_1, \dots, P_n\}$  be a (finite) family of finitely generated subgroups of  $G$ . The pair  $(G, \mathbb{P})$  is relatively hyperbolic if there is a generating set for  $G$  so that  $P_i \cap S$  generates  $P_i$  for each  $i \in \{1, \dots, n\}$  and that  $\text{Cus}(G, \mathbb{P}, S)$  is hyperbolic.*

The horoballings  $H_{\text{Cay}_{i,t}(S \cap P_i)} \subset \text{Cus}(G, \mathbb{P}, S)$ ,  $t \in G$ ,  $P_i \in \mathbb{P}$ , are horoballs and the action of  $G$  on  $\text{Cus}(G, \mathbb{P}, S)$  is cusp uniform. The cusped space provides us a way to define a boundary for relatively hyperbolic groups.

**Definition 3.8** (Bowditch boundary). *Given a finitely generated hyperbolic group  $G$  relative to a finite family of finitely generated subgroups  $\mathbb{P}$ , the Bowditch boundary  $\partial_{\mathbb{P}}G$  of  $(G, \mathbb{P})$  is defined as the boundary  $\partial \text{Cus}(G, \mathbb{P})$  of the cusped space.*

Corollary 3.17 below shows that the boundary is well defined, as a metric space, up to a quasi-Möbius change of metrics. Note that the topology of the boundary is well-defined according to [Bow6, Theorem 9.4]. Reference to a generating set is thus unnecessary.

In the next sections we will discuss quasi-isometries between relatively hyperbolic groups. In particular we will show that the previous definition does not depend on the choice of the generating set  $S$ .

**3.3.2. Geometrically finite actions.** Let  $G$  be a convergence group acting on some metrizable compact space  $Z$ . An element  $g \in G$  is *loxodromic* if it has infinite order and fixes exactly two points of  $Z$ . A subgroup  $H < G$  is *parabolic* if it is infinite and has a unique fixed point. We refer to it as a *parabolic point*. It contains no loxodromics [Bow5]. The stabilizer of a parabolic point is necessarily a parabolic group. There is thus a natural bijective correspondence between parabolic points in  $Z$  and maximal parabolic subgroups of  $G$ . We say that a parabolic group,  $H$ , with fixed point  $p$ , is *bounded* if the quotient  $(Z \setminus \{p\})/H$  is compact. (It is necessarily Hausdorff.) We say that  $p$  is a *bounded parabolic point* if its stabilizer is bounded. A *conical limit point* is a point  $y \in Z$  such that there exists a sequence  $(g_j)_{j \geq 0}$  in  $G$ , and distinct points  $a, b \in Z$ , such that  $g_j(y)$  tends to  $a$  and  $g_j(x)$  tends to  $b$  for all  $x \in Z \setminus \{y\}$ . We finally say that the action of  $G$  on  $Z$  is *geometrically finite* if every point of its limit set  $\Lambda_G$  is either conical or bounded parabolic (they cannot be both simultaneously). When  $G$  is a Kleinian group and  $Z = S^2$ , this definition is equivalent to the one given in §2.2.2, see for instance [Bow1] and the references therein.

When a relatively hyperbolic group  $(G, \mathbb{P})$  admits a cusp uniform action on a proper geodesic hyperbolic space  $X$  then its action on the boundary  $\partial X$  is geometrically finite, cf. [Bow6]. The conjugates of subgroups in  $\mathbb{P}$  are precisely the maximal parabolic subgroups. Conversely, if a group  $G$  admits a geometrically finite action on a metrizable space  $Z$ , then the pair  $(G, \mathbb{P})$  is relatively hyperbolic [Yam], where  $\mathbb{P}$  denotes a set of representatives of maximal parabolic subgroups of  $G$ . It turns out that the topology of  $\Lambda_G$  is independent of the space  $Z$  as long as the action is geometrically finite with the same parabolic subgroups [Bow6, Theorem 9.4]. See [Hru] for more information.

A subgroup  $H$  of a relatively hyperbolic group  $(G, \mathbb{P})$  is *elementary* if, whenever  $(G, \mathbb{P})$  admits a cusp uniform action on a proper geodesic hyperbolic space, the limit set of  $H$  has at most two points.

**3.4. Horoballs.** Let us set some facts about the metrics on the combinatorial horoballs  $H_P$  defined above.

**Fact 3.9.** *The distance on  $H_P$  has the following behavior:*

$$d \left[ \binom{x}{m}, \binom{y}{n} \right] = \max\{|m - n|, 2 \log_2(1 + d_P(x, y)) - (m + n)\} + O(1)$$

where  $d_P$  denotes the metric on  $P$ .

PROOF. By construction  $H_P$  is a geodesic space. We denote by  $\ell(\cdot)$  the length of a path, i.e., the number of edges it contains. Denote by  $\left[\binom{x}{n}, \binom{y}{n}\right]_h$  a horizontal path with minimal length and by  $\left[\binom{x}{m}, \binom{x}{n}\right]_v$  a vertical path.

It is easy to see that

$$(3.1) \quad d\left[\binom{x}{n}, \binom{x}{m}\right] = |m - n|$$

When  $x \neq y$ , we remark that:

$$\ell\left(\left[\binom{x}{n}, \binom{x}{n+1}\right]_v \cup \left[\binom{x}{n+1}, \binom{y}{n+1}\right]_h\right) \leq \ell\left(\left[\binom{x}{n}, \binom{y}{n}\right]_h \cup \left[\binom{y}{n}, \binom{y}{n+1}\right]_v\right)$$

Therefore, if  $\binom{x}{m}$  and  $\binom{y}{n}$  are two points in  $H_P$ , then they are joined by a geodesic of the form

$$\left[\binom{x}{m}, \binom{x}{k}\right]_v \cup \left[\binom{x}{k}, \binom{y}{k}\right]_h \cup \left[\binom{y}{k}, \binom{y}{n}\right]_v$$

with  $k \geq \max\{m, n\}$ . Hence

$$d\left[\binom{x}{m}, \binom{y}{n}\right] = \min_{k \geq \max\{m, n\}} \{2k - m - n + \lceil 2^{-k} d(x, y) \rceil\}$$

with  $\lceil a \rceil = p$  if  $p - 1 < a \leq p$  for some  $p \in \mathbb{Z}$ . When  $\max\{m, n\} \geq \log_2(d(x, y))$ , the minimum is reached for  $k = \max\{m, n\}$ :

$$(3.2) \quad d\left[\binom{x}{m}, \binom{y}{n}\right] = |m - n| + 1 \text{ when } \max\{m, n\} \geq \log_2(d(x, y))$$

When  $\max\{m, n\} < \log_2(d(x, y))$ , the minimum is reached for  $k = \lfloor \log_2(d(x, y)) \rfloor - 1$ :

$$(3.3) \quad d\left[\binom{x}{m}, \binom{y}{n}\right] = 2\lfloor \log_2(d(x, y)) \rfloor - (m + n) + f(d(x, y)) \text{ when } \max\{m, n\} < \log_2(d(x, y))$$

where  $0 \leq f(d(x, y)) = -2 + \lceil 2^{-\lfloor \log_2(d(x, y)) \rfloor + 1} d(x, y) \rceil \leq 2$ . We conclude the proof by combining equations (3.1), (3.2) and (3.3) with the fact that  $\ln(1 + a) = \max\{1, \ln a\} + O(1)$  for  $a \geq 0$ .  $\blacksquare$

We recall the following fact that we leave as an exercise for the reader.

**Fact 3.10.** *Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$ ,  $\lambda \geq 1$  and  $c, M \geq 0$  be such that*

- (1) *for all  $x, x' \in X$ ,  $d_Y(\varphi(x), \varphi(x')) \leq \lambda d_X(x, y) + c$  and for all  $y, y' \in Y$ ,  $d_X(g(y), g(y')) \leq \lambda d_Y(y, y') + c$ ;*
- (2) *for all  $x \in X$ ,  $d(g \circ f(x), x) \leq M$  and for all  $y \in Y$ ,  $d(f \circ g(y), y) \leq M$ .*

*Then  $f$  is a quasi-isometry.*



The following fact follows easily from the previous one.

**Fact 3.11.** *If a quasi-isometry  $G \rightarrow G'$  between finitely generated groups maps a finitely generated subgroup  $H < G$  at bounded Hausdorff distance from a finitely generated subgroup  $H' < G'$  then  $H$  and  $H'$  are quasi-isometric.*

Fact 3.10 is also used to extend quasi-isometric embeddings between metric spaces to their horoballings, see also [Grf, Lemma 6.2].

**Fact 3.12.** *Let  $f : P \rightarrow Q$  be a quasi-isometric embedding. Then  $F : H_P \rightarrow H_Q$  defined by*

$$F \begin{pmatrix} x \\ m \end{pmatrix} = \begin{pmatrix} f(x) \\ m \end{pmatrix}$$

*is a quasi-isometric embedding.*

PROOF. Let  $\lambda, c$  be such that  $\forall x, y \in P$ ,  $\lambda^{-1}d(f(x), f(y)) - c \leq d(x, y) \leq \lambda d(f(x), f(y)) + c$ .

Then we have:

$$\begin{aligned} d \left[ F \begin{pmatrix} x \\ m \end{pmatrix}, F \begin{pmatrix} y \\ n \end{pmatrix} \right] &\leq \max\{|m - n|, 2 \log_2(1 + \lambda d(x, y) + c) - (m + n)\} + O(1) \\ &\leq \max\left\{|m - n|, 2 \log_2 \left( \frac{1 + c}{\lambda} + d(x, y) \right) - (m + n)\right\} + O(1) \end{aligned}$$

If  $1 + c \leq \lambda$ , we have  $\log_2 \left( \frac{1 + c}{\lambda} + d(x, y) \right) \leq \log_2(1 + d(x, y))$ . Otherwise,

$$\begin{aligned} \log_2 \left( \frac{1 + c}{\lambda} + d(x, y) \right) &= \log_2 \left( \frac{1 + c}{\lambda} \right) + \log_2 \left( 1 + \frac{\lambda}{1 + c} d(x, y) \right) \\ &\leq \log_2(1 + d(x, y)) + \log_2 \left( \frac{1 + c}{\lambda} \right). \end{aligned}$$

In both cases, we get

$$d \left[ F \begin{pmatrix} x \\ m \end{pmatrix}, F \begin{pmatrix} y \\ n \end{pmatrix} \right] \leq d \left[ \begin{pmatrix} x \\ m \end{pmatrix}, \begin{pmatrix} y \\ n \end{pmatrix} \right] + O(1).$$

On the other hand, if  $1 + \lambda^{-1}d(x, y) - c > 0$ , we have

$$\begin{aligned} d \left[ F \begin{pmatrix} x \\ m \end{pmatrix}, F \begin{pmatrix} y \\ n \end{pmatrix} \right] &\geq \max\{|m - n|, 2 \log_2(1 + \lambda^{-1}d(x, y) - c) - (m + n)\} + O(1) \\ &\geq \max\{|m - n|, 2 \log_2((1 - c)/\lambda + d(x, y)) - (m + n)\} + O(1) \\ &\geq d \left[ \begin{pmatrix} x \\ m \end{pmatrix}, \begin{pmatrix} y \\ n \end{pmatrix} \right] + O(1). \end{aligned}$$

■

The following fact is then easily deduced from Facts 3.10 and 3.12:

**Fact 3.13.** *Let  $f : P \rightarrow Q$  be a quasi-isometry. Then  $F : H_P \rightarrow H_Q$  defined by*

$$F \begin{pmatrix} x \\ m \end{pmatrix} = \begin{pmatrix} f(x) \\ m \end{pmatrix}$$

*is a quasi-isometry. ■*

We now prove an analogous statement for horoballs in real hyperbolic space.

**Fact 3.14.** *Let  $P$  a graph endowed with the path metric so that each edge is isometric to  $[0, 1]$  and let  $\varphi : P \rightarrow (\mathbb{R}^d, d_E)$  be a quasi-isometric embedding. Consider a sequence  $\{y_n\} \in \mathbb{R}^{\mathbb{N}}$  such that  $1 \leq y_n \leq L$  for any  $n \in \mathbb{N}$  and some  $L > 0$  and define  $\Phi : H_P \rightarrow \mathbb{R}^d \times \mathbb{R}_+^* \approx \mathbb{H}^{d+1}$  by  $\Phi(x, n) = (\varphi(x), y_n 2^n)$ . Then  $\Phi$  is a quasi-isometric embedding. Furthermore, given  $Y \supset \varphi(P)$  and  $c$  such that a  $c$ -neighborhood of  $\varphi(P)$  covers  $Y$ , then  $\Phi : H_P \rightarrow Y \times [y_0, \infty)$  is a quasi-isometry.*

**PROOF.** We will use without notice the fact that  $\ln(1+x) = \max\{1, \ln x\} + O(1)$  for  $x \geq 0$ .

Note that in the upper half-space model,

$$\begin{aligned} d_{\mathbb{H}^{d+1}}((z, y), (z', y')) &= \operatorname{arcosh} \left( 1 + \frac{\|z - z'\|_E^2 + (y - y')^2}{2yy'} \right) \\ &= \ln \left( 1 + \frac{\|z - z'\|_E^2 + (y - y')^2}{2yy'} \right) + O(1) \\ &= \max \left\{ \ln \left( 1 + \frac{\|z - z'\|_E^2}{yy'} \right), \ln \left( 1 + \frac{(y - y')^2}{yy'} \right) \right\} + O(1) \\ &= \max \left\{ 2 \ln(1 + \|z - z'\|_E) - \ln yy', \left| \ln \left( \frac{y}{y'} \right) \right| \right\} + O(1). \end{aligned}$$

Therefore, since  $\{y_n\}$  is bounded,

$$\begin{aligned} d_{\mathbb{H}^{d+1}}((z, y_n 2^n), (z', y_m 2^m)) &= \max \{ 2 \ln(1 + \|z - z'\|_E) - (m+n) \ln 2, |m-n| \ln 2 \} + O(1) \\ &= \ln 2 \max \{ 2 \log_2(1 + \|z - z'\|_E) - (m+n), |m-n| \} + O(1). \end{aligned}$$

The rest of the proof follows as for Fact 3.12. ■

**3.4.1. Pared groups.** We adapt the notion of pared 3-manifolds to groups. This point of view was first developed by Otal for free groups in [Ota1].

Let  $G$  be a finitely generated group. A *paring* will be given by a finite *almost malnormal* collection of subgroups  $\mathbb{P}_G = \{P_1, \dots, P_k\}$ ; almost malnormal means that if  $P_i \cap (gP_jg^{-1})$  is infinite for some  $i, j \in \{1, \dots, k\}$  and  $g \in G$ , then  $i = j$  and  $g \in P_i$ . Most of the time,  $(G, \mathbb{P}_G)$  will be relatively hyperbolic and  $\mathbb{P}_G$  will be a set of representatives of conjugacy classes of maximal virtually Abelian groups such that any one-ended maximal virtually Abelian subgroup of  $G$  is conjugate to some  $P_i$ .

Note also that a pared compact 3-manifold  $(M, P_M)$  gives rise to a canonical pared group  $(K, \mathbb{P}_K)$  by letting  $K = \pi_1(M)$  and  $\mathbb{P}_K$  denote a representative for each conjugacy class of subgroups corresponding to the fundamental groups of the annuli and tori composing  $P_M$ . Let us call this pared group the *pared fundamental group* of  $(M, P_M)$ .

An isomorphism between pared groups  $(G, \mathbb{P}_G)$  and  $(Q, \mathbb{P}_Q)$  is an isomorphism  $G \rightarrow Q$  that maps the conjugacy classes in  $\mathbb{P}_G$  into the conjugacy classes in  $\mathbb{P}_Q$ .

A quasi-isometry between two pared groups  $(G, \mathbb{P}_G)$  and  $(H, \mathbb{P}_H)$  will be given by a quasi-isometry  $\Phi : G \rightarrow H$  such that the image of any coset of an element of  $\mathbb{P}_G$  is at bounded distance from a coset of an element of  $\mathbb{P}_H$ , and any coset of an element of  $\mathbb{P}_H$  is at bounded distance from the image of a coset of an element of  $\mathbb{P}_G$ . We will say that a pared manifold  $(M, P_M)$  is quasi-isometric to a pared group  $(G, \mathbb{P}_G)$  if there is a quasi-isometry between its pared fundamental group  $(K, \mathbb{P}_K)$  and  $(G, \mathbb{P}_G)$ .

**Fact 3.15** (induced paring). *Let  $(G, \mathbb{P})$  be a pared group and assume that  $H$  is a finite index subgroup of  $G$ . For every  $P \in \mathbb{P}$ , let  $T_P$  be a set of representatives of the double classes  $\{HgP, g \in G\}$  containing 1. The collection*

$$\mathbb{Q} = \{aPa^{-1} \cap H : P \in \mathbb{P}, a \in T_P\}$$

*defines a paring, called the induced paring of  $H$ , such that, for any  $g \in G$  and  $P \in \mathbb{P}$ , there exists  $h \in H$  and  $Q \in \mathbb{Q}$  such that  $\text{Stab}_H(gP) = hQh^{-1}$ .*

Observe that since  $H$  has finite index, this transversal is finite.

**PROOF.** Let us first show that  $\mathbb{Q}$  is a malnormal collection. Consider  $Q_1, Q_2 \in \mathbb{Q}$ ,  $h \in H$  and let us assume that  $hQ_1h^{-1} \cap Q_2$  is infinite. We may find  $P_1, P_2 \in \mathbb{P}$ ,  $a_1 \in T_{P_1}$  and  $a_2 \in T_{P_2}$  such that  $Q_j = a_jP_ja_j^{-1} \cap H$  for  $j = 1, 2$ . Thus

$$hQ_1h^{-1} \cap Q_2 = a_2[(a_2^{-1}ha_1)P_1(a_2^{-1}ha_1)^{-1} \cap P_2]a_2^{-1} \cap H$$

so the malnormality of  $\mathbb{P}$  implies that  $P_2 = P_1 = P \in \mathbb{P}$  and  $a_2^{-1}ha_1 \in P$ . Hence, we may find  $p \in P$  such that  $ha_1 = a_2p$ . By definition of  $T_P$ , this implies that  $a_1 = a_2 = a \in T_P$ . Hence  $Q_1 = Q_2 = Q$ . Furthermore, we have  $a^{-1}ha \in P$ , hence  $h \in aPa^{-1} \cap H = Q$ .

We have shown that  $\mathbb{Q}$  is a paring, let us look at the stabilizers of parabolic cosets. Let  $g \in G$  and  $P \in \mathbb{P}$ , there exist  $a \in T_P$ ,  $p \in \mathbb{P}$  and  $h \in H$  such that  $g = hap$ . Therefore

$$\text{Stab}_H(gP) = \text{Stab}_G(gP) \cap H = gPg^{-1} \cap H = (hap)P(hap)^{-1} \cap H = h(aPa^{-1} \cap H)h^{-1}.$$

■

**3.5. Quasi-isometries between pared groups.** Let  $(G, \mathbb{P})$  be a finitely generated pared group.

**Theorem 3.16.** *If two pared groups  $(G, \mathbb{P})$  and  $(G', \mathbb{P}')$  are quasi-isometric then the cusped spaces  $\mathcal{Cus}(G, \mathbb{P})$  and  $\mathcal{Cus}(G', \mathbb{P}')$  are also quasi-isometric.*

In the following proof, we actually don't use the malnormality of the paring.

**PROOF.** Let  $\varphi : (G, \mathbb{P}) \rightarrow (G', \mathbb{P}')$  be a quasi-isometry. Up to increasing the additive constant if necessary, we may assume that, for any  $g \in G$  and  $P \in \mathbb{P}$ , there are  $g' \in G'$  and  $P' \in \mathbb{P}'$  such that  $\varphi : gP \rightarrow g'P'$  is a quasi-isometry, cf. Fact 3.11.

Set  $\Phi : \mathcal{Cus}(G, \mathbb{P}) \rightarrow \mathcal{Cus}(G', \mathbb{P}')$  by letting  $\Phi = \varphi$  on  $\text{Cay}(G)$  and extending  $\varphi$  by Fact 3.12 to  $\Phi : H_{gP} \rightarrow H_{g'P'}$  for any  $g \in G$  and  $P \in \mathbb{P}$ . We note that each extension is a  $(\lambda, c)$ -quasi-isometry on the corresponding horoball, and that  $\varphi$  is also a  $(\lambda, c)$ -quasi-isometry for the graph metric of  $\text{Cay}(G)$ .

We conclude with a subdivision argument as in the proof of [Grf, Theorem 6.3]. Let  $x, x' \in \mathcal{Cus}(G, \mathbb{P})$  and let us consider a geodesic  $[x, x']$ . We may find a subdivision  $(x_j), 1 \leq j \leq p$ , of the geodesic segment such that each segment  $[x_j, x_{j+1}]$  is either contained in  $\mathcal{Cay}(G)$  (and may be degenerate) or in some  $H_{g_j P_{k_j}}$  (and have length at least 1). Notice that at least  $\lfloor \frac{p}{2} \rfloor$  of those segments lie in some  $H_{g_j P_{k_j}}$  and thus have length at least 1 hence  $p - 1 \leq 2d(x, x')$ . Then

$$d(\Phi(x), \Phi(x')) \leq \sum d(\Phi(x_j), \Phi(x_{j+1})) \leq \sum (\lambda d(x_j, x_{j+1}) + c) \leq (\lambda + 2c)d(x, x').$$

By symmetry, we obtain the same inequality for its quasi-inverse, hence Fact 3.10 concludes the proof (condition (2) is satisfied by construction since  $\Phi|_G : G \rightarrow G'$  and  $\Phi|_{H_{gP}} : H_{gP} \rightarrow H_{g'P'}$  are quasi-isometries).  $\blacksquare$

**Corollary 3.17.** *The topology and quasi-Möbius class of the boundary of the cusped space of a relatively hyperbolic group is independent from the choice of the generating set  $S$ .*

**Corollary 3.18.** *If a pared group is quasi-isometric to a relatively hyperbolic pared group, then it is hyperbolic relative to its paring.*

If  $K$  is a geometrically finite Kleinian group, then  $K$  is finitely generated, and relatively hyperbolic with respect to its maximal parabolic subgroups [Far, Theorem 5.1]. Let  $\mathbb{P}$  be a set of representatives of their conjugacy classes. We may then define  $\mathcal{Cus}(K)$  to be the cusped space of  $K$  by adding horoballs to the orbits of the elements of  $\mathbb{P}$ .

**Proposition 3.19.** *Let  $K$  be a geometrically finite Kleinian group and let  $\text{Hull}(\Lambda_K)$  be the convex hull of  $\Lambda_K$ . Pick a point  $o \in \text{Hull}(\Lambda_K)$ . Then the map  $k \mapsto k(o)$  extends to a quasi-isometry  $\Phi : \mathcal{Cus}(K) \rightarrow \text{Hull}(\Lambda_K)$  such that  $\Phi(\partial \mathcal{Cus}(K)) = \Lambda_K$ .*

PROOF. By assumption, we have  $\Phi(k) = k(o)$  for any  $k \in K$ . We extend  $\Phi$  to each horoball as in Fact 3.14.

Let  $U$  be a family of pairwise disjoint and  $K$ -invariant horoballs attached to the parabolic points of  $K$  such that  $o \notin U$ . Fix  $P \in \mathbb{P}$  and denote by  $z_P \in \widehat{\mathbb{C}}$  its parabolic fixed point. Represent  $\mathbb{H}^3$  by the upper half-space model  $\mathbb{R}^2 \times \mathbb{R}_+^*$  so that  $o$  becomes  $(0, 0, 1)$  and  $z_P$  corresponds to the point at infinity. Thus,  $\Phi(P) \subset \mathbb{R}^2 \times \{y_0\}$ , where  $y_0 = 1$ . Let  $y_1$  be large enough so that  $\mathbb{R}^2 \times \{2y_1\} \subset U$  and set  $y_n = y_1$  for  $n \geq 1$ . Let  $p : \mathbb{R}^2 \times \mathbb{R}_+^* \rightarrow \mathbb{R}^2$  be the projection along the third coordinate and define  $\Psi_P : H_P \rightarrow \mathbb{H}^3$  by  $\Psi_P(k, n) = (p \circ \Phi(k), y_n 2^n)$ . Now define  $\Phi$  on  $H_P$  by  $\Phi(k, n) = \Psi_P(k, n)$ . We do the same construction for each parabolic subgroup  $P \in \mathbb{P}$  and we extend the result equivariantly to get a map  $\Phi : \mathcal{Cus}(K) \rightarrow \text{Hull}(\Lambda_K)$ . The next Claim will conclude the proof of Proposition 3.19.  $\blacksquare$

**Claim.** — The map  $\Phi : \mathcal{Cus}(K) \rightarrow \text{Hull}(\Lambda_K)$  is a quasi-isometry.

PROOF. Up to taking smaller horoballs in the family  $U$ , we may assume that for any parabolic subgroup  $P$  and any  $k \in P$ ,  $\Phi(k, 1) \subset \partial U$ . Let  $K_+ \subset \mathcal{Cus}(K)$  be the union of  $K$  and the vertices at height 1 in the horoballs and denote by  $Y_K$  the maximal induced subgraph of  $\mathcal{Cus}(K)$  with vertex set  $K_+$ . We denote by  $Z_K$  the maximal subgraph of  $\mathcal{Cus}(K)$  whose vertices have height at least 1 in some horoball. Thus we have  $\mathcal{Cus}(K) = Y_K \cup Z_K$  and  $Y_K \cap Z_K$  is the maximal subgraph of  $\mathcal{Cus}(K)$  whose vertices have height exactly 1 in some horoball.

Since the actions of  $K$  on  $\overline{\text{Hull}(\Lambda_K)} \setminus U$  and on  $Y_K$  are cocompact, the restriction of  $\Phi$  to  $Y_K$  is a quasi-isometry between  $Y_K$  and  $\overline{\text{Hull}(\Lambda_K)} \setminus U$  endowed with the induced length metrics.

Let  $U_i$  be a horoball in the family  $U$  and represent  $\mathbb{H}^3$  by the upper half-space model  $\mathbb{R}^2 \times \mathbb{R}_+^*$  so that  $U_i = \{(x, y, z), z \geq 2y_1\}$ . Then  $\text{Hull}(\Lambda_K) \cap U_i$  has the form  $\{\text{Hull}(\Lambda_K) \cap \partial U_i\} \times [2y_1, \infty)$ . By the previous paragraph, the restriction of  $\Phi$  to  $Y_K \cap Z_K$  is a quasi-isometry to  $\text{Hull}(\Lambda_K) \cap \partial U$ . It follows then from Fact 3.14, that for any horoball  $H_{kP}$ , the restriction of  $\Phi$  to  $H_{kP} \cap Z_K$  is a quasi-isometry to the corresponding component of  $\text{Hull}(\Lambda_K) \cap U$ . The quasi-isometry constants are uniform since there are only finitely many orbits of parabolic points; let us denote them by  $(\lambda, c)$ .

In the previous paragraphs, we have seen that the restrictions  $\Phi|_{Y_K} : Y_K \rightarrow \overline{\text{Hull}(\Lambda_K)} \setminus Y$  and  $\Phi|_{Z_K} : Z_K \rightarrow \text{Hull}(\Lambda_K) \cap Y$  are quasi-isometries. It follows that there is  $D$  such that the  $D$ -neighbourhood of  $\Phi(\mathcal{Cus}(K))$  covers  $\text{Hull}(\Lambda_K)$ .

Pick  $k, k' \in \mathcal{Cus}(K)$ . With a subdivision as in the end of the proof of Theorem 3.16 we get:

$$d(\Phi(k), \Phi(k')) \leq 3 \max\{\lambda, c\}(d(k, k') + 1).$$

On the other hand, we may decompose  $[\Phi(k), \Phi(k')]$  into finitely many segments  $[x_j, x_{j+1}]$ ,  $1 \leq j < p$  so that  $x_1 = \Phi(k)$ ,  $x_p = \Phi(k')$ ,  $x_j \in \partial U$  for  $1 < j < p$  and  $]x_j, x_{j+1}[ \cap \partial U = \emptyset$  for  $1 \leq j < p$ . For each index  $1 < j < p$ , there is  $k_j \in Y_K \cap Z_K$  so that  $d(x_j, \Phi(k_j)) \leq c$  and we have  $\frac{1}{\lambda}d(k_j, k_{j+1}) \leq d(x_j, x_{j+1}) + 3c$  for any  $j < p$ , taking  $k_1 = k$  and  $k_p = k'$ . Let  $D > 0$  be the minimal distance between two horoballs in the family  $U$ . If  $]x_j, x_{j+1}[ \cap U = \emptyset$ , then  $d(x_j, x_{j+1}) \geq D$  and  $d(x_j, x_{j+1}) + 6c \leq (\frac{6c}{D} + 1)d(x_j, x_{j+1})$ . Since there are at least  $\lfloor \frac{p}{2} \rfloor$  such segments, we have:

$$\begin{aligned} d(k, k') &\leq \sum d(k_j, k_{j+1}) \leq \lambda \sum (d(x_j, x_{j+1}) + 3c) \\ &\leq 3c + \lambda \left( \frac{6c}{D} + 1 \right) \sum d(x_j, x_{j+1}) = \lambda \left( \frac{6c}{D} + 1 \right) d(\Phi(k), \Phi(k')) + 3c. \end{aligned}$$

■

**3.6. Quasiconvexity.** Let  $X$  be a proper, geodesic, hyperbolic metric space. A  $K$ -*quasiconvex* subset  $Y \subset X$  has the property that any geodesic segment joining two points of  $Y$  remains in the  $K$ -neighborhood of  $Y$ . Note that quasiconvexity is a property invariant under quasi-isometries.

Given a non trivial compact subset  $\Lambda \subset \partial X$ , we define its *weak convex hull* (or *join*)  $\mathcal{C}(\Lambda) \subset X$  as the union of all geodesics joining pair of points of  $\Lambda$ . Even though this set is usually not convex, it is shown in [KS] that it is uniformly quasiconvex. Note also that if  $\Lambda$  is a compact subset of  $\widehat{\mathbb{C}}$ , then the inclusion map  $\mathcal{C}(\Lambda) \hookrightarrow \text{Hull}(\Lambda)$  is a quasi-isometry in  $\mathbb{H}^3$ .

A subgroup  $H$  of a hyperbolic group  $G$  is *quasiconvex* if  $H$  is quasiconvex in any locally finite Cayley graph  $\text{Cay}(G)$  of  $G$ . A subgroup  $H$  of a relatively hyperbolic group  $(G, \mathbb{P})$  is *relatively quasiconvex* if, for any cusped space  $\mathcal{Cus}(G, \mathbb{P})$  for  $G$ , there is a constant  $L$  such for any geodesic  $\gamma \subset \mathcal{Cus}(G, \mathbb{P})$  with endpoints in  $H$ ,  $\gamma \cap \text{Cay}(G)$  remains at distance at most  $L$  from  $H$ .

Moreover, according to [Hru, Proposition 7.6], if  $H$  is a relatively quasiconvex subgroup of  $G$  and  $G$  has a cusp uniform action on  $\mathcal{Cus}(G, \mathbb{P})$ , then either  $H$  is finite or  $H$  is parabolic or the action of  $H$  on  $\mathcal{C}(\Lambda_H)$  is cusp uniform as well. In the latter case, the maximal parabolic subgroups of  $H$  define a finite number of  $H$ -conjugacy classes and are of the form  $gPg^{-1} \cap H$  where  $g \in G$ ,  $P \in \mathbb{P}$ , and such that  $gPg^{-1} \cap H$  is infinite. It follows that  $H$  is hyperbolic relative to representatives of these classes [Hru, Theorem 9.1].

**Proposition 3.20.** *Let  $(G_1, \mathbb{P}_1)$  and  $(G_2, \mathbb{P}_2)$  be two finitely generated relatively hyperbolic groups and let us consider two infinite, non parabolic and relatively quasiconvex finitely generated subgroups  $(H_1, \mathbb{Q}_1)$  and  $(H_2, \mathbb{Q}_2)$  of  $G_1$  and  $G_2$  respectively.*

*Let  $\varphi : (G_1, \mathbb{P}_1) \rightarrow (G_2, \mathbb{P}_2)$  be a quasi-isometry between pared groups together with its extension  $\Phi : \mathcal{Cus}(G_1, \mathbb{P}_1) \rightarrow \mathcal{Cus}(G_2, \mathbb{P}_2)$  given by Theorem 3.16 and denote by  $\partial\Phi : \partial\mathcal{Cus}(G_1, \mathbb{P}_1) \rightarrow \partial\mathcal{Cus}(G_2, \mathbb{P}_2)$  its boundary map. If  $\partial\Phi(\Lambda_{H_1}) = \Lambda_{H_2}$  then  $\varphi(H_1)$  is at bounded distance from  $H_2$  and  $(H_1, \mathbb{Q}_1)$  and  $(H_2, \mathbb{Q}_2)$  are quasi-isometric.*

We first analyse geometrically finite subgroups in the vicinity of its parabolic subgroups.

**Lemma 3.21.** *Let  $G$  be a discrete subgroup of isometries of a hyperbolic geodesic proper metric space  $X$ . Let  $H < G$  be a subgroup admitting a non-elementary action on  $\partial X$ . Let  $a \in \partial X$  be a parabolic point for  $H$ , and let  $Q = \text{Stab}_H a$  and  $P = \text{Stab}_G a$ . For any base point  $o \in X$  and any  $D > 0$ , there exists  $D'$  such that*

$$N_D(Ho) \cap N_D(Po) \subset N_{D'}(Qo),$$

where  $N_D(Y)$  denotes the  $D$ -neighborhood of a subset  $Y$ .

**PROOF.** Given  $o \in X$  and  $D > 0$  we are going to show that there are only finitely many elements  $h \in H$  such that  $h(o) \in N_D(Po)$  and  $h \notin Q$ .

For that purpose, consider infinite sequences  $(h_n)_n$  in  $H$  and  $(p_n)_n$  in  $P$  such that  $d(h_n(o), p_n(o)) \leq D$  and let us show that  $h_n \in Q$  for all but finitely many  $n$ 's.

We have  $d(p_n^{-1}h_n(o), o) \leq D$ , and since the action of  $G$  on  $X$  is properly discontinuous, there is a finite set  $F \subset G$  such that, for all  $n$ , we may find  $g \in F$  such that  $h_n = p_n g$ . We want to show that  $g \in P$  for all  $g \in F$  for which  $g = p_n^{-1}h_n$  for infinitely many  $n$ 's. We are thus led to the situation where  $g \in G$ ,  $(p_n)_n$  is an infinite sequence of  $P$  and  $h_n = p_n g$  with  $h_n \in H$ .

We first use the fact that the action of  $G$  on  $X \cup \partial X$  is a convergence action [Bow5, Prop. 1.12]. Therefore, by [Bow5, Prop. 1.1], since  $(p_n)_n$  is an infinite sequence of the parabolic group  $P$ , we have convergence of  $(p_n^{\pm 1}(o))_n$  to  $a$ . Thus, we also have convergence of  $(h_n^{\pm 1}(o))_n$  to  $a$  since its distance to  $(p_n^{\pm 1}(o))_n$  is bounded.

But since  $h_n = p_n g$ , the sequence  $(h_n^{-1}(o))_n$  tends to  $g^{-1}(a)$ , so that  $g^{-1}(a) = a$  and  $g \in P$ . Therefore,  $h_n \in P \cap H = Q$  for all but finitely many  $n$ 's. ■

**PROOF OF PROPOSITION 3.20.** Let  $j = 1, 2$  and let us denote by  $\mathcal{Cay}_j$  a locally finite Cayley graph of  $G_j$  used to build the cusped space  $\mathcal{Cus}_j = \mathcal{Cus}(G_j, \mathbb{P}_j)$ . As mentioned above, the action of  $H_j$  on  $\mathcal{C}(\Lambda_{H_j})$  ( $\subset \mathcal{Cus}_j$ ) is cusp uniform. Hence the action of  $H_j$  on the truncated space  $\mathcal{C}_T(H_j) = \mathcal{C}(\Lambda_{H_j}) \cap \mathcal{Cay}_j$  is cocompact. It follows that the Hausdorff distance between  $H_j \subset \mathcal{Cus}_j$  and  $\mathcal{C}_T(H_j)$  is bounded since they are both  $H_j$ -invariant.

By the shadowing lemma (Lemma 3.2),  $\Phi(\mathcal{C}(\Lambda_{H_1}))$  is quasiconvex, at bounded distance from  $\mathcal{C}(\Lambda_{H_2})$ . By assumption,  $\Phi(\mathcal{Cay}_1) = \varphi(\mathcal{Cay}_1)$  is at bounded distance from  $\mathcal{Cay}_2$ . It follows then from the previous paragraph that  $\Phi(\mathcal{C}_T(H_1))$  is at bounded distance from  $\mathcal{C}_T(H_2)$  as well, implying that  $\Phi(H_1)$  is at bounded distance from  $H_2$  in  $\mathcal{Cus}_2$ . Let us note that the canonical injection  $\mathcal{Cay}_j \hookrightarrow \mathcal{Cus}_j$ ,  $j = 1, 2$ , is uniformly continuous by Fact 3.9, so we may conclude that  $\varphi(H_1)$  is at bounded distance from  $H_2$  in  $\mathcal{Cay}_2$ .

It remains to prove that the quasi-isometry preserves the parings.

Let  $Q_1 \in \mathbb{Q}_1$  be a parabolic subgroup of  $H_1$  with parabolic point  $p$ . There are  $g_1 \in G_1$  and  $P_1 \in \mathbb{P}_1$  such that  $Q_1 = g_1 P_1 g_1^{-1} \cap H_1$ . By assumption, we may find  $P_2 \in \mathbb{P}_2$  and  $g_2 \in G_2$  such that  $\varphi(g_1 P_1)$  is at bounded distance from  $g_2 P_2$ . Let us consider  $Q_2 = g_2 P_2 g_2^{-1} \cap H_2$ . Since  $\varphi(p)$  is a parabolic for  $G_2$ , it is also parabolic for  $H_2$  since  $H_2$  is geometrically finite, so  $Q_2$  is infinite.

Let us observe that, for  $j = 1, 2$ ,  $g_j P_j g_j^{-1}$  is parabolic, so preserves horospheres centered at  $p$  or  $\varphi(p)$ , and  $g_j P_j$  is also a horosphere centered at the same point, so  $g_j P_j$  and  $g_j P_j g_j^{-1}$  lie at bounded distance. On the one hand,  $\varphi(g_1 P_1)$  is at bounded distance from  $g_2 P_2$ , and, on the other hand,  $\varphi(H_1)$  is at bounded distance from  $H_2$ . It follows that  $\varphi(Q_1)$  is at bounded distance from  $H_2$  and from  $g_2 P_2$ , so from  $g_2 P_2 g_2^{-1}$ . But  $H_2 \cap g_2 P_2 g_2^{-1} = Q_2$ , so Lemma 3.21 now implies that  $\varphi(Q_1)$  is at bounded distance from  $Q_2$ . Therefore,  $\varphi$  maps the peripheral structure for  $H_1$  into a bounded neighborhood of the peripheral structure of  $H_2$ . By symmetry, we conclude that  $(H_1, \mathbb{Q}_1)$  and  $(H_2, \mathbb{Q}_2)$  are quasi-isometric as pared groups. ■

Let  $(G, \mathbb{P}_G)$  be a relatively hyperbolic group and let  $\mathbb{P} \supset \mathbb{P}_G$  be a paring for  $G$  by infinite relatively quasiconvex subgroups. Define on the Bowditch boundary  $\partial_{\mathbb{P}_G} G = \partial \mathcal{Cus}(G, \mathbb{P}_G)$  an equivalence relation  $\sim_{\mathbb{P}}$  as follows: let  $x \sim_{\mathbb{P}} y$  if, either  $x = y$  or if there is a subgroup  $P \in \mathbb{P}$  and an element  $g \in G$  such that  $\{x, y\} \subset g(\Lambda_P)$ . Set  $Q_{\mathbb{P}} = \partial_{\mathbb{P}_G} G / \sim_{\mathbb{P}}$  to be the quotient of  $\partial_{\mathbb{P}_G} G$  by this relation  $\sim_{\mathbb{P}}$ .

Being unable to find the following generalization of [Bow6, Theorem 7.11] in the literature, we sketch a proof of it.

**Proposition 3.22.** *With the above notation,  $(G, \mathbb{P})$  is relatively hyperbolic and there is a  $G$ -equivariant homeomorphism between  $Q_{\mathbb{P}}$  and the Bowditch boundary of  $(G, \mathbb{P})$ .*

We start with the following lemma.

**Lemma 3.23.** *Let  $(G, \mathbb{P}_G)$  be a relatively hyperbolic group and let  $H$  be a non elementary relatively quasiconvex subgroup such that  $\mathbb{P}_G \cup \{H\}$  forms an almost malnormal family. The following properties hold.*

- (1) *The action of  $H$  is uniform on  $\Lambda_H$  and cocompact on  $\partial_{\mathbb{P}_G} G \setminus \Lambda_H$ .*
- (2) *We have  $H = \text{Stab}_G \Lambda_H$ .*
- (3) *The collection of compact sets  $\mathcal{K} = \{g \Lambda_H \mid g \in G\}$  forms a null sequence of pairwise disjoint compact sets, i.e., for any distance on  $\partial_{\mathbb{P}_G} G$ , for any  $\delta > 0$ , there are only finitely many sets in  $\mathcal{K}$  with diameter at least  $\delta$ .*

**PROOF.** Let  $X = \mathcal{Cus}(G, \mathbb{P}_G)$ , and let us consider the action of  $G$  on  $X \cup \partial X$ . We note that  $H$  has no parabolic elements. If this was the case, then there would be some  $g \in G$  and  $P \in \mathbb{P}_G$  such that  $g P g^{-1} \cap H$  would be infinite, which contradicts the almost malnormality

assumption. Therefore, since  $H$  is relatively quasiconvex, this implies that every point in  $\Lambda_H$  is conical. We may then deduce that the action of  $H$  is uniform on  $\Lambda_H$  by [Bow3, Thm. 8.1] and cocompact on  $\partial_{\mathbb{P}_G} G \setminus \Lambda_H$  by [Swe, Main Thm. (3)]. Furthermore, by the corollary of the main theorem in [Swe],  $H$  has finite index in  $\text{Stab}_G \Lambda_H$ . If  $g \in \text{Stab}_G \Lambda_H$ , then  $gHg^{-1} \cap H$  is a finite index subgroup of  $H$ , hence infinite, so that  $g \in H$  by malnormality. We have proved (1) and (2).

By the corollary to [Swe, Thm. 13], we also know that  $g\Lambda_H \cap \Lambda_H = \Lambda_{gHg^{-1} \cap H}$  so that either  $g \in H$  or  $g\Lambda_H \cap \Lambda_H = \emptyset$ . This shows that the elements of  $\mathcal{K}$  are pairwise disjoint. Let us fix  $\delta > 0$ . There exists  $R > 0$  such that  $d(e, gH) \leq R$  whenever  $\text{diam} g\Lambda_H \geq \delta$  since  $H$  is quasiconvex in  $X$  [Swe, Main Thm. (1)]. This implies that there are only finitely many such elements  $g \in G/H$ . This concludes the proof. ■

**PROOF OF PROP. 3.22.** We proceed in two steps. We first establish that  $\sim_{\mathbb{P}}$  defines an upper semi-continuous decomposition of  $\partial_{\mathbb{P}_G} G$ , i.e., the equivalence relation  $\sim_{\mathbb{P}}$  is closed. This implies that  $Q_{\mathbb{P}}$  is Hausdorff and compact. Then we prove that the action of  $G$  on  $Q_{\mathbb{P}}$  is geometrically finite with the prescribed parabolic subgroups.

By Lemma 3.23, the limit sets  $\{g\Lambda_P, P \in \mathbb{P}, g \in G\}$  form a null sequence of pairwise disjoint sets, so they define an upper semi-continuous decomposition of  $\partial_{\mathbb{P}_G} G$ . This shows that the quotient  $Q_{\mathbb{P}}$  is a Hausdorff compact space, and the group  $G$  acts on  $Q_{\mathbb{P}}$ .

Lemma 3.23 also implies (a) that the action of any  $P \in \mathbb{P} \setminus \mathbb{P}_G$  on  $\partial_{\mathbb{P}_G} G \setminus \Lambda_P$  is cocompact, so they define bounded parabolic groups on  $Q_{\mathbb{P}}$  and (b) that they are maximal parabolic subgroups.

Let us now check that all the other points are conical. If we consider such a point  $z$  with preimage  $x \in \partial_{\mathbb{P}_G} G$ , then we may find distinct points  $\alpha, \beta \in \partial_{\mathbb{P}_G} G$  and a sequence of elements  $(g_n)$  such that  $(g_n(x))_n$  tends to  $\alpha$  while all the other sequences  $(g_n(y))_n$  tend to  $\beta$ . If  $\alpha$  and  $\beta$  lie in different fibers of the projection map  $\pi : \partial_{\mathbb{P}_G} G \rightarrow Q_{\mathbb{P}}$ , then it follows that  $z$  is also conical. If they belong to a common fiber, they belong to some  $g\Lambda_P, P \in \mathbb{P} \setminus \mathbb{P}_G, g \in G$  and we may as well assume that  $g$  is trivial. As  $x \notin \Lambda_P$  and the action of  $P$  on  $\partial_{\mathbb{P}_G} G \setminus \Lambda_P$  is cocompact, we may find  $(h_n)_n$  in  $P$  so that  $(h_n g_n)(x)$  tends to a point  $a \in \partial_{\mathbb{P}_G} G \setminus \Lambda_P$ . By [Bow5, Prop. 1.1], we may assume that  $(h_n)_n$  has the convergence property, i.e., tends to a point  $b \in \Lambda_P$  uniformly on compact subsets of  $\partial_{\mathbb{P}_G} G \setminus \{\alpha\}$  (since  $(g_n(x))_n$  tends to  $\alpha$ ). Since  $\beta \neq \alpha$ , it follows and  $(h_n g_n(y))$  tends to  $b$  for all  $y \neq x$ . Thus the limits of  $(h_n g_n(x))$  and  $(h_n g_n(y))_m$  will be in different fibers for all  $y \neq x$ . Therefore, we may also conclude that  $z$  is conical.

In conclusion, we have defined a geometrically finite action on  $Q_{\mathbb{P}}$  with the prescribed maximal parabolic subgroups. Thus  $(G, \mathbb{P})$  is relatively hyperbolic and by definition,  $Q_{\mathbb{P}}$  is equivariantly homeomorphic to the boundary of the cusped space  $\partial \mathcal{Cus}(G, \mathbb{P})$ . ■

#### 4. CANONICAL SPLITTINGS

We describe well-known splittings of manifolds and their counterparts for finitely presented groups and show the quasi-isometric invariance of those splittings. This provides the first step in the proof of Theorems 1.2 and 1.4 as described in §1. We start with the definition of a graph of groups structure and see how they appear when splitting manifolds.



Let us recall that our main results are concerned with groups up to quasi-isometry, so up to finite-index. In particular, since any manifold admits a finite covering of degree at most two that is orientable, we may —and will always— assume that our manifolds are orientable.

**4.1. Graph of groups and manifold splittings.** Let  $G$  be a group. We first recall the definition of a graph of groups and set up some notations. We follow Serre, and define a graph as a pair of sets  $(V, E)$ , with a fixed-point free involution  $e \rightarrow \bar{e}$  on  $E$  (exchanging an oriented edge with its reverse orientation), and a terminal map  $t : E \rightarrow V$  mapping an edge to the vertex it is oriented to.

A *graph of groups*  $\mathcal{G} = (\Gamma, \{G_v\}, \{G_e\}, G_e \hookrightarrow G_{t(e)})$  is

- a graph  $\Gamma = (V, E)$ ;
- an assignment of a group  $G_e$  or  $G_v$  to each edge  $e$  or vertex  $v$  of  $\Gamma$ , satisfying  $G_{\bar{e}} = G_e$ ; and
- for each edge  $e \in E$ , a monomorphism  $G_e \hookrightarrow G_{t(e)}$ .

A graph of groups as above defines a group  $G$  up to isomorphism called the fundamental group of the graph of groups [Ser, §5]. It is characterized by an action on a simplicial tree  $T$ , called the *Bass-Serre tree* of the graph of groups, with the following properties. The action is minimal, with no edge inversions and the orbit space  $T/G$  is isomorphic to  $\Gamma$ . Moreover, for any vertex  $v \in T$ ,  $\text{Stab}(v)$  is isomorphic to  $G_{p(v)}$ , for any edge  $e \in T$ ,  $\text{Stab}(e)$  is isomorphic to  $G_{p(e)}$  and the canonical injection  $\text{Stab}(e) \subset \text{Stab}(t(e))$  projects to the injection  $G_{p(e)} \hookrightarrow G_{t(p(e))}$ , where  $p : T \rightarrow \Gamma = T/G$  denotes the canonical projection.

A *graph of groups structure* for a group  $G$  is a graph of groups  $\mathcal{G} = (\Gamma, \{G_v\}, \{G_e\}, G_e \hookrightarrow G_{t(e)})$  together with an isomorphism between  $G$  and the fundamental group of  $\mathcal{G}$  as defined above.

The structure is *finite* if  $\Gamma$  is finite and is *trivial* if there is a vertex group equal to  $G$ . Unless otherwise stated, graph of groups structures will be assumed to be finite and non trivial. We say that  $G$  *splits* over a subgroup  $H$  (which can be trivial) if there is a graph of groups structure for  $G$  in which  $H$  is an edge group.

Let us first notice that a splitting of a compact 3-manifold  $M$  produces a graph of groups structure for its fundamental group as follows, see [ScW] for details.

Let  $M$  be a compact 3-manifold and let  $S$  be a finite collection of disjoint non isotopic essential surfaces. We define the tree  $\mathcal{T}_S$  dual to  $S$  as follows: vertices are lifts of the components of  $M \setminus S$  to  $\widetilde{M}$  and there is an edge between two vertices if the closures of the corresponding components intersect. The action of  $\pi_1(M)$  on  $\widetilde{M}$  by covering transformations induces an action of  $\pi_1(M)$  on  $\mathcal{T}_S$  by isometries. This action provides  $Q = \pi_1(M)$  with a graph of groups structure  $\mathcal{G} = (\Delta, \{Q_v\}, \{Q_e\}, Q_e \hookrightarrow Q_{t(e)})$ :  $\Delta = \mathcal{T}_S/Q$ ; edge groups are fundamental groups of components of  $S$  and vertex groups are fundamental groups of components of  $M \setminus S$ .

The components of  $M \setminus S$  are not compact and their closures might give a different decomposition. To remedy this we set up the following definition:

**Definition 4.1** (Submanifold compactification). *Let  $N$  be a component of  $M \setminus S$  and  $\widetilde{N}$  a lift of  $N$  to  $\widetilde{M}$ . We define the compactification  $\bar{N}$  of  $N$  as the quotient of the closure of  $\widetilde{N}$  under the action of its stabilizer in  $\pi_1(M)$  (which is isomorphic to  $\pi_1(N)$ ).*

**4.2. Splittings over finite groups.** Let  $M$  be a compact 3-manifold and let  $S \subset M$  be an essential surface. As explained above, the action of  $\pi_1(M)$  on the dual tree  $\mathcal{T}_S$  gives rise to a graph of groups structure to  $\pi_1(M)$ . If  $S$  is a union of pairwise non isotopic spheres and discs, then the edge groups are trivial and if furthermore  $S$  is maximal, the vertex groups are fundamental groups of irreducible and boundary irreducible 3-manifolds.

With the next proposition, we show how to reverse this construction starting from a graph of groups with vertex groups corresponding to irreducible 3-manifolds.

**Proposition 4.2.** *Let  $G$  be a group with a graph of groups structure*

$$\mathcal{G} = (\Gamma, \{G_v\}, \{G_e\}, G_e \hookrightarrow G_{t(e)})$$

*with finite graph  $\Gamma$  and with finite edge groups. Assume that each vertex group  $G_v$  has a finite index normal subgroup  $G'_v$  isomorphic to the fundamental group of an irreducible orientable compact 3-manifold  $M_v$  (with or without boundary), then  $G$  is commensurable to the fundamental group of a compact 3-manifold  $M$ .*

*Furthermore, if  $\partial M_v$  is non-empty for every vertex  $v$ , then  $M$  is irreducible. If  $M_v$  is atoroidal and  $\chi(M_v) < 0$  for all  $v$ , then we may choose  $M$  atoroidal and irreducible.*

**PROOF.** If  $G_v$  is finite, we may as well assume that  $G'_v$  is trivial. Otherwise, since  $M'_v$  is orientable and irreducible,  $G'_v$  is torsion free for any vertex  $v$  [Hat, Prop. 2.45]. For an edge  $e = (v, w)$ , since  $G_e$  is finite,  $G'_v \cap G_e$  and  $G'_w \cap G_e$  are trivial. Consider the graph of groups

$$\overline{\mathcal{G}} = (\Gamma, \{\overline{G}_v\}, \{\overline{G}_e\}, \overline{G}_e \hookrightarrow \overline{G}_{t(e)})$$

where  $\overline{G}_v = G_v/G'_v$  and  $\overline{G}_e = \{1\}$ . Let  $\overline{G}$  be the fundamental group of  $\overline{\mathcal{G}}$ , which is a finite graph of finite groups, hence is virtually free [ScW, Thm. 7.3] and residually finite [Sta2].

The canonical projections  $G_v \rightarrow \overline{G}_v$  define a projection  $q : G \rightarrow \overline{G}$  such that for any vertex  $v$ ,  $\ker q \cap G_v = G'_v$ . Since  $\overline{G}$  is residually finite, there is a finite index subgroup  $\overline{K}$  which is disjoint from any non trivial element of any vertex group  $\overline{G}_v$ . Let  $\overline{Q} = \bigcap_{g \in \overline{G}} g \overline{K} g^{-1}$  be the normal core of  $\overline{K}$  in  $\overline{G}$  (which has finite index in  $\overline{G}$ ) and let  $p : \overline{G} \rightarrow \overline{G}/\overline{Q}$  be the canonical projection. By construction, the kernel  $Q$  of  $p \circ q$  is a finite index normal subgroup of  $G$  such that  $G_v \cap Q = G'_v$  for any vertex group  $G_v$ . It follows that  $Q$  has a graph of groups structure  $(\Gamma', \{G'_v\})$  with trivial edge groups.

For each edge  $e = (v, w)$  of  $\Gamma'$  we proceed as follows: if  $M_v$  and  $M_w$  have non-empty boundaries we pick a disc on  $\partial M_v$  and a disc on  $\partial M_w$  and glue  $M_v$  to  $M_w$  along those discs (we may have  $M_v = M_w$ ). If  $M_v$  or  $M_w$  has no boundary, we remove a ball from  $M_v$  and a ball from  $M_w$  and glue  $M_v$  to  $M_w$  along the resulting boundary spheres. We do this operation for every edge by choosing disjoint discs and balls. It is easy to deduce from van Kampen's theorem that the fundamental group of the resulting manifold  $M$  has a graph of groups structure  $(\Gamma', \{G'_v\})$  with trivial edge groups [ScW]. Hence  $\pi_1(M) = Q$  and we are done.

Notice that if every manifold  $M_v$  has non-empty boundary, then we have only glued along discs so  $M$  is irreducible.

Assume that every manifold  $M'_v$  has non-empty boundary and that  $M$  contains an essential torus  $T$ . If  $T$  is not contained in a manifold  $M_v$  then it intersects an essential disc  $D$ . Up to isotopy, we may assume that  $T$  and  $D$  are transverse. Since  $T$  is incompressible, any component of  $T \cap D$  bounds a disc  $D_1$  in  $T$ . Since  $M$  is irreducible,  $D_1$  is isotopic in  $M$  to a

disc in  $D$ . It follows that  $T$  can be changed by an isotopy to be disjoint from  $D$ . Thus we have proved that a component of  $M_v$  contains either an essential torus or a torus in its boundary. Therefore, if all vertex manifolds  $M_v$  are also atoroidal, so is  $M$ . ■

We already mentioned the fact that a compact 3-manifold  $M$  is irreducible and has incompressible boundary if and only if  $\pi_1(M)$  is one-ended. Thus, when we decompose  $M$  along a maximal union of essential spheres and discs, we get a graph of groups with trivial edge groups and one-ended or finite vertex groups.

A similar decomposition has been established by Stallings for finitely generated groups [Sta1]. If  $G$  is non-elementary and not one-ended, it splits over a finite group, i.e.,  $G$  is the fundamental group of a graph of groups with finite edge groups. Such a graph of groups is called *terminal* if it is finite and the vertex groups are one-ended or finite. A group which has a terminal splitting is *accessible*. According to Dunwoody [Dun], finitely presented groups are accessible.

Terminal graph of groups are invariant under quasi-isometries [PW], see below. Combining Proposition 4.2 and Theorem 4.3, it suffices to prove Theorem 1.2 and 1.4 for one-ended groups.

**Theorem 4.3** (Papasoglu & Whyte [PW]). *Let  $G$  be an accessible group and let*

$$\mathcal{G} = (\Gamma, \{G_v\}, \{G_e\}, G_e \hookrightarrow G_{t(e)})$$

*be a terminal graph of groups decomposition of  $G$ . A group  $G'$  is quasi-isometric to  $G$  if and only if it is also accessible and any terminal decomposition of  $G'$  has the same set of quasi-isometry types of one-ended factors and the same number of ends.*

**4.3. Torus decomposition.** The second splitting of  $M$  that we will use is related to its characteristic torus decomposition. The torus decomposition together with the annulus decomposition (see §4.4) compose the JSJ splitting of  $M$ . For more on the history of the next statement due to Johannson and Jaco-Shalen [JS, Joh], see [Bon, Theorem 3.4].

**Theorem 4.4** (Characteristic torus decomposition). *Let  $M$  be an orientable compact irreducible 3-manifold. Then, up to isotopy, there is a unique compact 2-dimensional submanifold  $T$  of  $M$  with the following properties.*

- (i) *Every component of  $T$  is an essential torus.*
- (ii) *The compactification of every component of  $M \setminus T$  either contains no essential embedded torus or else admits a Seifert fibration.*
- (iii) *Property (ii) fails when any component of  $T$  is removed.*

We call  $T$  the *characteristic torus decomposition* of  $M$ . It follows from the works of Thurston and Perel'man that  $T$  is also a geometric decomposition, in particular any manifold with an empty characteristic torus decomposition is geometric. Conversely geometric manifolds have empty characteristic torus decompositions except the quotients of  $Sol$  [Sco3, Thm 5.3]. Thus the uniqueness statement in Theorem 4.4 implies the uniqueness (up to isotopy) of the geometric decomposition if we add a minimality assumption in the spirit of property (iii).

As explained at the beginning of §4,  $T$  induces on  $\pi_1(M) = Q$  a structure of graph of groups  $\mathcal{G} = (\Delta, \{Q_v\}, \{Q_e\}, Q_e \hookrightarrow Q_{t(e)})$  where edge groups are isomorphic to  $\mathbb{Z}^2$  and vertex groups are fundamental groups of components of  $M \setminus T$ .

To make full use of the work of Kapovich and Leeb, we will use a smaller decomposition (with larger pieces). Let  $M$  be an irreducible non-geometric 3-manifold and  $T$  be the characteristic torus decomposition of  $M$ . The *Euler characteristic decomposition*  $T_{Eu} \subset T$  is the union of the components of  $T$  which bound a component of  $M \setminus T$  with negative Euler characteristic.

We would like to have a similar splitting for a group quasi-isometric to  $\pi_1(M)$ . Different versions of the JSJ splitting have been given for finitely generated groups (see [GL] for a survey) but they tend to be different from the one given by the torus decomposition above since  $\pi_1(M)$  may also split over cyclic subgroups. Rather than using this general theory, we will follow [KaL3] and use the quasi-isometry to construct a splitting of  $G$ .

**Theorem 4.5** (Quasi-isometric invariance of the torus decomposition). *Let  $G$  be a finitely generated group quasi-isometric to the fundamental group of a non-geometric irreducible  $\partial$ -irreducible orientable 3-manifold  $M$ . Let  $S = T$  or  $S = T_{Eu}$  be the characteristic torus decomposition or the Euler characteristic decomposition of  $M$ .*

*The group  $G$  has a graph of groups structure  $\mathcal{G} = (\Gamma, \{G_v\}, \{G_e\}, G_e \hookrightarrow G_{t(e)})$  and there is a map  $i : \Gamma^{(0)} \rightarrow \{\text{Components of } M \setminus S\}$  such that:*

- for each vertex  $v$ ,  $G_v$  is quasi-isometric to  $\pi_1(W_{i(v)})$  for some component  $W_{i(v)}$  of  $M \setminus S$ ;
- two vertices  $v, w$  are adjacent if and only if  $W_{i(v)}$  and  $W_{i(w)}$  are adjacent;
- if  $W_{i(v)}$  has zero Euler characteristic, then  $G_v$  has a finite index subgroup  $G'_v$  which is the fundamental group of a compact 3-manifold  $M'_v$  with zero Euler characteristic and for any adjacent edge  $e = (v, w)$ ,  $G'_v \cap G_e$  is conjugate to the fundamental group of a boundary component of  $M'_v$ .

When  $M$  has zero Euler characteristic this result follows from the work of Kapovich and Leeb [KaL3] and Theorem 1.3. In the general case the conclusion can still be deduced from the arguments of [KaL3] and Theorem 1.3 as we will explain now.

The starting point of Kapovich and Leeb's proof is that  $M$  is nonpositively curved in the large [KaL1, Theorem 1.1]. More precisely, if  $M$  has zero Euler characteristic, there exist a nonpositively curved compact 3-manifold of  $N$  and a bi-Lipschitz homeomorphism between their universal covers  $\widetilde{M}$  and  $\widetilde{N}$  that preserves their torus decompositions.

For manifolds with negative Euler characteristic, we can start with a stronger statement, using the arguments of [Lee, Theorems 3.2 and 3.3].

**Proposition 4.6.** *Let  $M$  be an irreducible  $\partial$ -irreducible 3-manifold with non-empty boundary. Then  $M$  admits a Riemannian metric with nonpositive curvature.*

PROOF. We will recall some of the arguments of the proof of [Lee, Theorem 3.3] to show that it easily extends to manifolds with negative Euler characteristic.

If  $M$  is geometric, it is either hyperbolic or Seifert fibered, cf. [Sco3, §4], in particular Theorems 4.3, 4.13, 4.15, 4.16 and 4.17. In the latter case,  $M$  has a finite ramified cover  $M'$  which is a circle bundle over a compact surface  $F$  [Hem1, Theorem 12.2]. Since  $\partial M \neq \emptyset$ ,  $\partial F \neq \emptyset$ ,  $M'$  is a trivial bundle and it admits a  $\mathbb{H}^2 \times \mathbb{E}^1$  structure which projects to  $M$ . It follows that  $M$  is modelled on  $\mathbb{H}^3$  or  $\mathbb{H}^2 \times \mathbb{E}^1$  and the conclusion is obvious.

If  $M$  is not geometric, let  $T$  be the torus decomposition of  $M$ . Denote by  $M_H$  the union of the atoroidal components of  $M \setminus T$  and denote by  $M_G$  the closure of  $M \setminus \overline{M}_H$ . By construction,

$M_G$  is a graph manifold with non-empty boundary and, by (the proof of) [Lee, Theorem 3.2],  $M_G$  admits a Riemannian metric with nonpositive curvature and with flat boundary. By the hyperbolization theorem, the interior of  $M_H$  admits a complete hyperbolic metric. Now, notice that the proof of [Lee, Proposition 2.3] consists in changing the metric of hyperbolic 3-manifolds only in the rank 2 cusps. Following exactly the same proof, we get that any flat metric on  $T$  can be extended to a nonpositively curved metric on  $M_H$ . In particular, we can extend the nonpositively curved metric on  $M_G$  to a metric on  $M$  with nonpositive curvature. ■

Thus we may assume that  $M$  is equipped with a Riemannian metric with nonpositive curvature and that  $S$  is a union of totally geodesic flat tori. To continue the proof of Proposition 4.5, we need to set up a few definitions.

Suppose that  $G$  is a group and  $\rho$  is a map from  $G$  to the set of all  $(K, \epsilon)$ -quasi-isometries of a metric space  $X$ . We call  $\rho$  a *quasi-action* of  $G$  if for some constant  $L$  and all  $g_1, g_2 \in G$  the quasi-isometries  $\rho(g_1 g_2)$  and  $\rho(g_1) \circ \rho(g_2)$  are  $L$ -close. The quasi-action is called *quasi-transitive* if for some constant  $M$  all orbits  $\rho(G).x$  are  $M$ -close to  $X$ . The *kernel* of the action  $\rho$  is the subgroup of  $G$  which consists of elements whose action on  $X$  is Hausdorff-close to the identity. A quasi-action is called *properly discontinuous* if for each bounded subset  $C \subset X$  there are only finitely many elements  $g_j \in G$  so that  $\rho(g_j)(C) \cap C \neq \emptyset$ .

We say that a collection  $\mathcal{A}$  of subsets  $A \subset X$  is *quasi-invariant* under the quasi-action  $\rho$  if:

- every bounded subset  $B \subset X$  intersects only finitely many sets in  $\mathcal{A}$ ;
- any two distinct sets in  $\mathcal{A}$  have infinite Hausdorff distance;
- there is a constant  $H$  such that for all  $g \in G$  and  $A \in \mathcal{A}$  the set  $\rho(g)(A)$  is  $H$ -Hausdorff close to another set in  $\mathcal{A}$ .

We can define the *stabilizer* in  $G$  of a set  $A$  in a quasi-invariant collection  $\mathcal{A}$ . It consists of all elements  $g \in G$  such that  $\rho(g)(A)$  and  $A$  have finite Hausdorff distance (recall that  $A$  is unbounded by definition). Clearly the stabilizer is a subgroup of  $G$  and it is easy to define its quasi-action on  $A$ . This quasi-action is properly discontinuous and quasi-transitive if the quasi-action of  $G$  on  $X$  has such properties, see [KaL3, Lemma 5.2].

PROOF OF THEOREM 4.5. By Proposition 4.6,  $M$  may be endowed with a metric of nonpositive curvature. The group  $G$  is quasi-isometric to its universal cover  $\widetilde{M}$ . Let  $f_1 : \widetilde{M} \rightarrow G$  and  $f_2 : G \rightarrow \widetilde{M}$  be quasi-isometric embeddings such that  $d(f_2 \circ f_1(x), x) \leq C$  and  $d(f_1 \circ f_2(y), y) \leq C$  for some  $C \geq 0$  and for any  $x \in \widetilde{M}$  and any  $y \in G$ . Given  $g \in G$ , it is easy to see that the map  $\rho(g) : \widetilde{M} \rightarrow \widetilde{M}$  defined by  $\rho(g)(x) = f_2(g f_1(x))$  is a quasi-isometry and that  $\rho$  is a quasi-transitive properly discontinuous quasi-action of  $G$  on  $\widetilde{M}$  with finite kernel. We deduce the following lemma from the work of Kapovich and Leeb.

**Lemma 4.7.** *If  $S = T$  or  $S = T_{Eu}$  and  $\widetilde{S} \subset \widetilde{M}$  denote the preimage of  $S$ , then the collections of connected components of  $\widetilde{S}$  and  $\widetilde{M} \setminus \widetilde{S}$  are quasi-invariant under  $\rho$ .*

PROOF. This lemma follows from [KaL3, Proposition 3.11], [KaL3, Theorem 4.6] and [KaL3, Lemma 4.7]. Notice that these results hold in our slightly more general context. ■

PROOF OF THEOREM 4.5 (CONTINUED). Let  $\mathcal{T}_S$  be the tree dual to  $\tilde{S}$ , as defined in §4.1. By Lemma 4.7,  $\rho(g)$  induces an automorphism  $\sigma(g)$  on  $\mathcal{T}_S$ . Since  $\rho$  is a quasi-action,  $\sigma(g_1)\sigma(g_2) = \sigma(g_1g_2)$ , and we get a simplicial action of  $G$  on  $\mathcal{T}_S$ . The quotient  $\Gamma = \mathcal{T}_S/G$  is finite and the action induces a graph of groups structure  $\mathcal{G} = (\Gamma, \{G_v\}, \{G_e\}, G_e \hookrightarrow G_{t(e)})$  for  $G$  where vertex and edge groups are stabilizers under the quasi-action  $\rho$  of components of  $\tilde{M} \setminus \tilde{S}$  and  $\tilde{S}$  respectively.

Let  $G_v$  be the stabilizer of a component  $X$  of  $\tilde{M} \setminus \tilde{S}$ . Since  $G_v$  acts quasi-transitively on  $X$ , then  $G_v$  is quasi-isometric to  $X$  which is the universal cover of a component  $W_{i(v)}$  of  $M \setminus S$ . It follows that  $G_v$  is quasi-isometric to  $\pi_1(W_{i(v)})$ .

By construction to every edge  $e = (v, w)$  is associated a component of  $S$  and two vertices  $v, w$  are adjacent if and only if  $W_{i(v)}$  and  $W_{i(w)}$  are adjacent.

This proves the two first properties of  $\mathcal{G} = (\Gamma, \{G_v\}, \{G_e\}, G_e \hookrightarrow G_{t(e)})$ .

When  $S = T$  all components of  $M - S$  are geometric and the third property follows from Theorems 2.7 and 1.3.

When  $S = T_{Eu}$  we need to go deeper into the arguments of [KaL3] and explain how to glue the geometric pieces together. Let  $X$  be a component of  $\tilde{M} \setminus \tilde{T}$  bounded by quasi-flats, equivalently  $X$  is the universal cover of a component  $W$  of  $M \setminus T$  with zero Euler characteristic. The stabilizer  $G_X < G$  of  $X$  is conjugate to a vertex group  $G_v$ . By [Sch] and [Rie], see also [KaL3, §5.2],  $G_X$  fits into a short exact sequence

$$1 \rightarrow \text{Fin}(G_X) \rightarrow G_X \rightarrow H_X \rightarrow 1$$

where  $\text{Fin}(G_X)$  is the kernel of the quasi-action of  $G_X$  on  $X$ —hence is finite since the quasi-action is properly discontinuous—and  $H_X$  is the fundamental group of a compact 3-orbifold  $O_X$  with flat boundary. In particular  $H_X$  has no finite subgroup and  $\text{Fin}(G_v)$  is the unique maximal finite normal subgroup of  $G_X$ . The stabilizer  $G_E < G_X$  of a component  $E$  of  $\tilde{T}$  bounding  $X$  is conjugate to the vertex group  $G_e$  of an edge  $e = (v, w)$  adjacent to  $v$  and we have a short exact sequence

$$1 \rightarrow \text{Fin}(G_X) \rightarrow G_E \rightarrow H_E \rightarrow 1$$

where  $H_e$  is the fundamental group of a closed Euclidean 2-orbifold. Hence  $\text{Fin}(G_X)$  is also the unique maximal finite normal subgroup of  $G_E$ . It follows that  $\text{Fin}(G_X)$  is also the kernel of the quasi-action of  $G_Y$  on the component  $Y$  of  $\tilde{M} \setminus \tilde{T}$  lying on the other side of  $E$  if  $Y$  is bounded by quasi-flats.

Let  $Z$  be a component of  $\tilde{M} \setminus \tilde{T}_{Eu}$  bounded by quasi-flats and let  $G_Z < G$  be its stabilizer. We have defined for  $G_Z$  a graph of groups structure  $\mathcal{G}_Z = (\Gamma_Z, \{G_v\}, \{G_e\}, G_e \hookrightarrow G_{t(e)})$  for  $\tilde{G}$ . By the previous paragraph the unique maximal finite normal subgroups of all vertex and edge stabilizers coincide and therefore coincide with the kernel  $\text{Fin}(G_Z)$  of the quasi-action of  $G_Z$  on  $\tilde{Z}$ . For each vertex  $v \in \Gamma_Z$ ,  $G_v/\text{Fin}(G_Z)$  is the fundamental group of a 3-dimensional orbifold  $O_v$  with flat boundary and for any adjacent edge group  $e = (v, w)$ ,  $G_e/\text{Fin}(G_Z)$  is conjugate to the fundamental group of a boundary component of  $O_v$ . Gluing these orbifolds  $G_v$  along boundary components according to the graph  $\Gamma_Z$  yields an orbifold  $O_Z$  with fundamental group  $G_Z/\text{Fin}(G_Z)$ . By [MM],  $G_Z/\text{Fin}(G_Z)$  has a finite index subgroup which is the fundamental

group of a 3-manifold with zero Euler characteristic. The conclusion follows from Theorem 1.3. ■

**4.4. Annulus decomposition.** Lastly, we will introduce the JSJ decomposition along annuli for atoroidal 3-manifolds and a generalization to relatively hyperbolic groups. This will lead to splittings of groups quasi-isometric to Kleinian groups with the following properties.

**Theorem 4.8** (Geometric decomposition of quasi-Kleinian one-ended groups). *Let  $G$  be a finitely generated one-ended group quasi-isometric to a minimally parabolic geometrically finite Kleinian group  $K$ . Then  $G$  has a graph of groups structure  $\mathcal{G} = (\Gamma, \{G_v\}, \{G_e\}, G_e \hookrightarrow G_{t(e)})$  with the following properties:*

- (1) *edge groups are virtually cyclic;*
- (2) *edge groups incident to an Abelian vertex group are all commensurable;*
- (3) *Abelian vertex groups are virtually cyclic or virtually  $\mathbb{Z}^2$ ;*
- (4) *Abelian vertex groups are not adjacent to each other nor to themselves;*
- (5) *for every non-Abelian vertex group  $G_v$ , there is a pared compact hyperbolic 3-manifold with pared fundamental group  $(H_v, \mathbb{P}_v)$  and a quasi-isometry between  $(H_v, \mathbb{P}_v)$  and  $G_v$  equipped with the paring provided by adjacent edges and parabolic subgroups.*

To prove that proposition, we will build a splitting of  $K$  in two different ways. On the one hand, we will introduce the second part of the JSJ decomposition of the Kleinian manifold  $M_K$ : the characteristic annulus decomposition. This decomposition is obtained by doubling the manifold along its boundary and considering the restriction to the initial manifold of the torus decomposition of the double. Thus we get a family  $A$  of annuli that cuts the manifold into pieces with some specific topological properties. As explained at the beginning of §4.1 the action of  $K = \pi_1(M_K)$  on the dual tree  $\mathcal{T}_A$  to  $A$  induces a graph of groups structure for  $K$ .

On the other hand, following [Bow2] and [PS], we will build a tree  $\mathcal{T}_{\Lambda_K}$  from the topological features of the limit set  $\Lambda_K$ . We will show that  $\mathcal{T}_{\Lambda_K}$  is isomorphic to  $\mathcal{T}_A$  and that the action of  $K$  on  $\Lambda_K$  induces the action of  $K$  on  $\mathcal{T}_A = \mathcal{T}_{\Lambda_K}$ . We will then see that the quasi-isometry between  $G$  and  $K$  extends to a homeomorphism between their Bowditch boundaries. This will yield an action of  $G$  on  $\mathcal{T}_{\Lambda_K}$  and hence a splitting of  $G$ . Finally the quasi-isometric invariance of those splittings will provide us with the desired properties (1) to (5).

**4.4.1. Annulus decomposition and JSJ tree.** We start by introducing the characteristic annulus decomposition, which, together with the torus decomposition described in Section 4.3, form the JSJ decomposition defined by Johannson-Jaco-Shalen [Joh, JS], see also [Bon, Theorem 3.8]. As explained above it can be defined using the torus decomposition of the manifold obtained by doubling  $M$  along its boundary, but it will be convenient to have a more straightforward definition.

**Theorem 4.9** (Characteristic annulus decomposition). *Let  $(M, P)$  be an orientable compact atoroidal boundary irreducible pared 3-manifold. Then, up to isotopy, there is a unique compact 2-dimensional submanifold  $(A, \partial A) \subset (M, \partial M \setminus P)$  of  $M$  such that:*

- (i) *Every component of  $A$  is an essential annulus.*

- (ii) *The compactification of every component of  $M \setminus A$ , is either pared acylindrical or a pared  $I$ -bundle or a solid torus or a thickened torus.*
- (iii) *Property (ii) fails when any component of  $A$  is removed.*

We call  $A$  the *characteristic annulus decomposition* of  $M$ . Notice that the compactification  $\overline{W}$  of each component  $W$  of  $M \setminus A$  inherits a paring  $P_W$  coming from  $P$  and  $A$  defined by  $P_W = (P \cap W) \cup (\partial \overline{W} \setminus (W \cap \partial M))$ . A *pared  $I$ -bundle* is a pared manifold  $(N, P)$  such that  $N$  is homeomorphic to a product  $F \times I$  over a compact surface  $F$  by a homeomorphism that maps  $P$  into  $\partial F \times I$ .

The action of  $\pi_1(M)$  on the dual tree to  $A$  induces a graph of groups structure for  $\pi_1(M)$ . As previously mentioned, we will give an alternate construction of this action when  $M$  is uniformized by a Kleinian group. To simplify the identification of the two constructions we add solid tori to  $M \setminus A$  so that a component that is pared acylindrical or a pared  $I$ -bundle is only adjacent to solid tori and thickened tori. Concretely we replace each component  $A_i$  of  $A$  that does not lie in the closure of any solid torus or thickened torus component of  $M \setminus A$  by two disjoint parallel copies of itself, thus adding a solid torus to  $M \setminus A$ . We call the resulting surface  $B$  the *balanced annulus decomposition* of  $M$ , it has the following property:

- (ii') *Every component of  $B$  lies in the closure of a component of  $M \setminus A$  whose compactification is a solid torus or a thickened torus.*

Notice that if we set

- (iii') *Property (ii) or (ii') fails when any component of  $B$  is removed.*

then the balanced annulus decomposition is uniquely defined (up to isotopy) by properties (i), (ii), (ii') and (iii').

To give an alternate definition of  $\mathcal{T}_B$  using the limit set  $\Lambda_K$  of a Kleinian group  $K$  uniformizing  $M$ , we will now, following [PS], define the subsets of  $\Lambda_K$  that will be used as vertices. Notice that since  $M$  is assumed to be boundary irreducible,  $\Lambda_K$  is connected.

Given a continuum (i.e., a connected compact set)  $X$ , a point  $x \in X$  is a *cut point* if  $X \setminus \{x\}$  is not connected. We define an equivalence relation  $\mathcal{R}$  on  $\Lambda_K$ . Each cut point is only equivalent to itself and if  $a, b \in \Lambda_K$  are not cut points we say that  $a \mathcal{R} b$  if they are not separated by any cut point, i.e. for any cut point  $c \in \Lambda_K$ ,  $a$  and  $b$  lie in the same component of  $\Lambda_K \setminus \{c\}$ . By [PS, Lemma 32], when  $X$  is a Peano (i.e., locally connected) continuum the closure  $Y$  of each non singleton equivalence class is a Peano continuum without cut points.

Let  $Y \subset X$  be the closure of a non singleton equivalence class for  $\mathcal{R}$ . A pair  $\{a, b\} \subset Y$  is a *cut pair* if  $Y \setminus \{a, b\}$  is not connected. A nondegenerate nonempty subset  $A \subset Y$  is called *inseparable* if no two points of  $A$  lie in different components of the complement of any cut pair. Every inseparable set is contained in a maximal inseparable set.

A finite subset  $S$  of a  $Y$  is called a *cyclic subset* if either  $S$  is a cut pair or there is an ordering  $S = \{s_j, j \in \mathbb{Z}/n\mathbb{Z}\}$ ,  $n \geq 3$ , and continua  $M_j \subset Y$ ,  $j \in \mathbb{Z}/n\mathbb{Z}$ , such that

- $M_i \cap M_{i+1} = \{s_i\}$ ,  $i \in \mathbb{Z}/n\mathbb{Z}$ ,
- $M_i \cap M_j = \emptyset$  whenever  $|i - j| > 1$ ,
- $\bigcup M_i = Y$ .



An infinite subset in which all finite subsets of cardinality at least 2 are cyclic is also called *cyclic*. A maximal cyclic subset with at least 3 elements is called a *necklace*.

In [PS], the authors use these subsets to construct an  $\mathbb{R}$ -tree associated to a Peano continuum called the *combined tree*. We introduce a simplified version of this construction when the continuum is the limit set  $\Lambda_K$  of a geometrically finite Kleinian group  $K$ . Consider the set  $V$  of cut points, cut pairs, necklaces and maximal inseparable sets of  $\Lambda_K$  (cut pairs, necklaces and maximal inseparable sets are taken in the closures of non singleton equivalence classes of the relation  $\mathcal{R}$  defined above). As we will see in Propositions 4.11 and 4.16, this set  $V$  is countable. We define a graph  $\mathcal{T}_{\Lambda_K}$  with vertex set  $V$  by putting an edge between two vertices  $v_1, v_2$  if  $v_1 \subset v_2$  as subsets of  $\Lambda_K$ . Rather than showing that this graph is a simplicial tree, we will directly show that it is isomorphic to the dual tree  $\mathcal{T}_B$  to the balanced annulus decomposition of the Kleinian manifold  $M_K$ .

We conclude this subsection with a remark for the reader familiar with the work of [PS]. We will see in Proposition 4.16 that the closure of a necklace intersects a maximal inseparable set only along an inseparable cut pair. Using [PS, Lemma 8 and Lemma 28] and the definition of  $T$  in [PS, p. 1765], one can see that  $\mathcal{T}_{\Lambda_K}$  is the *combined tree* associated to  $\Lambda_K$  defined on [PS, p. 1782].

In the next sections we will describe the stabilizers of the sets constituting  $V$  and their relation with the characteristic annulus decomposition.

**4.4.2. Cut points stabilizers.** In this section we establish some properties of stabilizers of cut points. They will be used to prove that  $\mathcal{T}_{\Lambda_K}$  is isomorphic to  $\mathcal{T}_B$  as well as properties (1) and (2) of Theorem 4.8. For the latter we will work in the general situation of relatively hyperbolic groups acting on their Bowditch boundary.

**Proposition 4.10.** *Let  $(G, \mathbb{P})$  be a relatively hyperbolic group and assume that its Bowditch boundary  $\partial_{\mathbb{P}}G$  is connected and locally connected. Let  $p \in \partial_{\mathbb{P}}G$  be a cut point and let us denote by  $\mathcal{C}$  the collection of connected components of  $X = \partial_{\mathbb{P}}G \setminus \{p\}$ . Then  $p$  is a bounded parabolic point the stabilizer of which we denote by  $H$ . The action of  $H$  on  $\mathcal{C}$  has finitely many orbits and, for any  $C \in \mathcal{C}$ , the action of  $\text{Stab}_H C$  is cocompact on  $C$ , hence infinite. Moreover, any compact subset of  $X$  meets at most finitely many components of  $X$ .*

**PROOF.** The fact that cut points are bounded parabolic points is due to Bowditch [Bow4, Thm 0.2].

Let  $L$  be a compact fundamental domain for the action of  $H$  on  $X$ . Every point in  $L$  admits a connected neighborhood in  $X$ , hence contained in a unique element of  $\mathcal{C}$ . By compactness, this implies that only finitely many components of  $\mathcal{C}$  intersect  $L$ . This shows that  $\mathcal{C}$  is a finite union of  $H$ -orbits. This also implies that any compact subset  $K$  of  $X$  meets at most finitely many components of  $X$ , since the fact that the action is properly discontinuous on  $X$  implies that there are only finitely many elements of  $H$  that maps points of  $K$  in  $L$ .

Let us denote by  $C_1, \dots, C_n \in \mathcal{C}$  the components that intersect  $L$ . If  $C_i$  and  $C_j$  are in the same orbit, we fix  $h_{ij} \in H$  such that  $h_{ij}(C_j) = C_i$ , with  $h_{ii} = id$ . Given  $C = C_i$ , write  $L_i = \cup h_{ij}(C_j \cap L)$  where the union is taken over the components  $C_j$  in the same  $H$ -orbit than  $C_i$ . Note that  $L_i$  is a compact subset of  $C_i$ , as a finite union of compact subsets of  $C_i$ . If  $x \in C_i$ , then we may find  $g \in H$  such that  $g(x) \in L$  by definition. If  $g(x) \in C_j$ , then  $h_{ij}(gx) \in L_i$ .

Since  $h_{ij}g(C_i) = C_i$  by construction, we have proved that the action of  $\text{Stab}_H C_i$  is cocompact on  $C_i$ . Since  $C_i$  is non-compact (the point  $p$  is an accumulation point by definition), we may conclude that its stabilizer is infinite. ■

A parabolic isometry  $g$  in a Kleinian group  $K$  is an *accidental parabolic* if  $g$  stabilizes a component  $\mathcal{O}$  of the domain of discontinuity and a geodesic in  $\mathcal{O}$  equipped with its hyperbolic metric.

**Proposition 4.11.** *Let  $K$  be a geometrically finite Kleinian group with connected limit set. A point  $p \in \Lambda_K$  is a cut point if and only if its stabilizer contains a primitive accidental parabolic.*

PROOF. By Selberg's lemma, we may assume that  $K$  is torsion free. Since  $p$  is a cut point, one can find two disjoint closed sets  $A$  and  $B$  of  $X = \Lambda_K \setminus \{p\}$  which covers  $X$ . Note that, on  $\widehat{\mathbb{C}}$ ,  $\overline{A} \cap \overline{B} = \{p\}$ . Therefore, by the separation theorem [Why1, Thm VI.3.1], we may find a Jordan curve  $c$  that separates a point  $a \in A$  from a point  $b \in B$  such that  $c \cap \Lambda_K = \{p\}$ .

By Proposition 4.10, the point  $p$  is parabolic; denote by  $H$  its stabilizer (which is isomorphic to  $\mathbb{Z}$  or  $\mathbb{Z}^2$ ).

Let  $\mathcal{O}$  be the component of  $\Omega_K$  that contains the connected set  $c \setminus \{p\}$  and  $K_{\mathcal{O}}$  be its stabilizer. Ahlfors finiteness theorem implies that  $\mathcal{O}/K_{\mathcal{O}}$  is a surface of finite type. Moreover, Thurston proved that  $K_{\mathcal{O}}$  is geometrically finite as well, since it is finitely generated [Mor, Theorem 7.1]. Since  $p$  is not conical for  $K$ , it cannot be conical for  $K_{\mathcal{O}}$  either, so it is parabolic. This implies that  $K_{\mathcal{O}} \cap H$  is a cyclic group, generated by an element  $h$ .

The curve  $c$  disconnects  $\partial\mathcal{O}$  since, otherwise,  $c$  would bound a disc in  $\mathcal{O}$ , and, hence, would not separate  $A$  and  $B$ . By Proposition 4.10, the stabilizer  $K_C$  in  $K_{\mathcal{O}}$  of a component  $C$  of  $\partial\mathcal{O} \setminus \{p\}$  is cyclic. Each element in  $K_C$  fixes the two distinct ends of  $C$ , hence the hyperbolic geodesic  $\gamma$  in  $\mathcal{O}$  joining them. This implies that  $p$  is an accidental parabolic.

Conversely, let  $\gamma \subset \Omega_K$  be a hyperbolic geodesic stabilized by a parabolic isometry  $h \in K$ . Then  $\gamma$  joins the fixed point  $p$  of  $h$  to itself and  $\gamma \cup p$  is Jordan curve separating  $\Lambda_K$ . It follows that  $p$  is a cut point.

If  $h$  is not primitive, then there exists  $g \in K$  and an iterate  $k \geq 1$  such that  $g^k = h$ . We first observe that  $g \in H$  since  $g^k$  fixes  $p$ . Let us note that the closure of  $\gamma$  is a Jordan curve, so, if  $g$  does not fix  $\gamma$  then the latter is mapped within one of the complementary component of  $\gamma$ . But  $g$  being parabolic, it acts as a translation so none of its powers can map  $\gamma$  to itself again, a contradiction. Therefore,  $h$  may be chosen to be primitive in  $K$ . ■

**Corollary 4.12.** *Let  $K$  be a geometrically finite Kleinian group with connected limit set. Assume that the point  $p$  at infinity is a cut point with stabilizer isomorphic to  $\mathbb{Z}^2$ . Then  $X = \Lambda_K \cap \mathbb{C}$  has infinitely many components with common stabilizers that are cyclic. Moreover, given a component  $C$  of  $\Lambda_K \cap \mathbb{C}$  and  $R > 0$ , there are only finitely many other components at Euclidean distance at most  $R$  from  $C$ .*

PROOF. Let  $H$  be the stabilizer of  $p$ ; let us consider a component  $\mathcal{O}$  that is fixed by an infinite subgroup of  $H$ , and let  $H_{\mathcal{O}} = H \cap \text{Stab}_K \mathcal{O}$ . It follows from Proposition 4.11 that  $H_{\mathcal{O}}$  is a cyclic group generated by a primitive element  $h_1$ . Thus, we may choose another primitive element  $h_2 \in H$ , different from  $h_1^{\pm 1}$ . It follows that  $h_1$  and  $h_2$  generate  $H$ .

Moreover,  $h_2$  acts freely on the  $H$ -orbit of  $\mathcal{O}$ , and so on the components of  $\Lambda_K \cap \mathbb{C}$  since the  $H$ -orbit of  $\mathcal{O}$  splits the plane in parallel (topological) strips.

Let  $C$  be a component of  $\Lambda_K \cap \mathbb{C}$ , and assume that  $h \in H$  fixes  $C$ . Then we may find iterates  $k, \ell \in \mathbb{Z}$  such that  $h = h_1^k h_2^\ell$ . It follows from above that  $\ell = 0$  and  $h$  is an iterate of  $h_1$ . This shows that  $\text{Stab}_H C \subset H_{\mathcal{O}}$ . Conversely,  $h_1(C) = C$  since it fixes every element of  $H_{\mathcal{O}}$ ; implying that at least one iterate will fix  $C$ . Thus, every component  $C$  has stabilizer the subgroup generated by  $h_1$ , i.e.,  $H_{\mathcal{O}}$ .

**Claim.**— The sequence  $(h_2^n)_n$  is uniformly convergent to  $p$  on any component  $C$  of  $X$ .

If not, there would be a sequence of points  $(x_k)$  in  $C$  such that  $h_2^{n_k}(x_k)$  remains in a compact subset  $K$  of  $X$  for some subsequence  $(n_k)$ . Since the action of  $\text{Stab}_H C$  is cocompact on  $C$ , we may assume that  $(x_k)$  remains in a compact subset of  $C$ . Since the action is properly discontinuous, this implies that an iterate of  $h_2$  fixes  $C$ , which is absurd. This proves the claim.

Fix  $R > 0$  and a component  $C$  of  $\Lambda_K \cap \mathbb{C}$ . Fix another component  $C'$ , and let assume that there are  $m$  translates of  $C'$  at distance at most  $R$  from  $C$ . We may find  $(n_k)$ ,  $1 \leq k \leq m$ , such that  $\text{dist}(C, h_2^{n_k} C') \leq R$  for all  $k$ . Since the actions of the stabilizers of components are cocompact on their components, we may assume that these distances are realized in some compact  $L_R$  of  $X$ . The claim implies that  $m$  has a uniform upper bound. So there are only finitely many components at distance at most  $R$  from  $C$ . ■

**Lemma 4.13.** *Let  $K$  be a Kleinian group with connected limit set  $\Lambda_K$  and let  $G$  be a group virtually isomorphic to  $\mathbb{Z}^2$ , acting by uniformly quasi-Möbius maps on  $\Lambda_K$ . We assume that the action of  $G$  is bounded parabolic with common fixed point a cut point  $p \in \Lambda_K$  and that the stabilizer of  $p$  in  $K$  has rank 2. Then every stabilizer in  $G$  of a connected component of  $\Lambda_K \setminus \{p\}$  is virtually cyclic, and they are pairwise commensurable.*

*Moreover, there exists a finite index free Abelian subgroup  $G_A$  generated by  $g_1, g_2 \in G$  such that  $g_1$  stabilizes each connected component and generates a subgroup acting cocompactly on each component and  $g_2$  acts freely on the components of  $\Lambda_K \setminus \{p\}$ .*

**PROOF.** We may assume that the point  $p$  is at infinity in  $\widehat{\mathbb{C}}$  so that  $G$  acts by uniform quasisymmetric maps on  $X = \mathbb{C} \cap \Lambda_K$ . This means that there exists an increasing homeomorphism  $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that, for any  $x, y, z \in X$  and any  $g \in G$ ,

$$\text{if } |x - y| \leq t|x - z| \text{ then } |g(x) - g(y)| \leq \eta(t)|g(x) - g(z)|.$$

Since  $p$  is a cut point,  $X$  has at least 2 connected components. Let  $H \subset K$  denote the stabilizer of  $p$ . By assumption, it is a rank 2 Abelian group acting properly discontinuously by translations on  $\mathbb{C}$ . By Corollary 4.12 there are constants  $\delta_{\pm} > 0$  such that, for any component  $C$  of  $X$ ,  $\text{dist}(C, X \setminus C) \geq \delta_-$  and for any  $x \in C$ ,  $\text{dist}(x, X \setminus C) \leq \delta_+$  hold.

We first prove that we may deform the Euclidean metric on  $X$  quasisymmetrically so that  $G$  acts by isometries. Let  $x, y \in X$ . We claim that  $d(x, y) = \sup\{|g(x) - g(y)|, g \in G\}$  is finite. Let  $C_x$  be the component of  $X$  containing  $x$ . Given  $g \in G$ , pick  $z \in X \setminus C_x$  so that  $|g(x) - g(z)| = \text{dist}(g(x), X \setminus g(C_x))$ , then we have:

$$|g(x) - g(y)| \leq \eta\left(\frac{|x - y|}{|x - z|}\right) |g(x) - g(z)| \leq \eta\left(\frac{|x - y|}{\delta_-}\right) \delta_+.$$

This implies that  $(X, d)$  is a metric space on which  $G$  acts by isometries. Moreover, for any  $x, y, z \in X$  and  $\varepsilon > 0$ , if  $g \in G$  satisfies  $d(x, y) \leq (1 + \varepsilon)|g(x) - g(y)|$ , then

$$\frac{d(x, y)}{d(x, z)} \leq (1 + \varepsilon) \frac{|g(x) - g(y)|}{|g(x) - g(z)|} \leq (1 + \varepsilon) \eta \left( \frac{|x - y|}{|x - z|} \right).$$

Since  $\varepsilon$  is arbitrary, it follows that  $id : (X, |\cdot|) \rightarrow (X, d)$  is  $\eta$ -quasisymmetric.

Since the action of  $G$  is bounded parabolic, Proposition 4.10 implies that there are only finitely many orbits of such connected components. Moreover, their stabilizer are virtually cyclic since  $G$  has rank two and there are infinitely many components, cf. Corollary 4.12.

If  $C'$  is another component, then, for any  $g \in \text{Stab}_G C$ ,  $\text{dist}(C, g(C')) = \text{dist}(C, C')$  so that the orbit of  $C'$  under  $\text{Stab}_G C$  is finite by Corollary 4.12. This implies that  $\text{Stab}_G C \cap \text{Stab}_G C'$  has finite index in  $\text{Stab}_G C$ . Therefore, stabilizers are commensurable.

Let  $G'$  be a rank two free Abelian subgroup of finite index in  $G$ . Then stabilizers of components in the same  $G'$ -orbit are the same and cyclic. They also act cocompactly on each of them. Therefore, there are only finitely many such stabilizers and their intersection is a cyclic group generated by some  $g_1$  of finite index in any stabilizer. Let us consider a primitive  $g_2 \in G'$  so that it generates with  $g_1$  a subgroup  $G_A$  of rank two. The element  $g_2$  acts freely on the components of  $X$  by construction. ■

4.4.3. *Stabilizers and characteristic annulus decomposition.* Next we will study the relations between the stabilizers of the sets that make up  $V$  and the balanced annulus decomposition. We start with the cut points.

**Lemma 4.14.** *Let  $K$  be a minimally parabolic geometrically finite Kleinian group uniformizing a pared manifold  $(M, P)$  with associated representation  $\rho : \pi_1(M) \rightarrow K$ . Let  $B$  be the balanced annulus decomposition of  $(M, P)$ . If the limit set  $\Lambda_K$  is connected, then a point  $c \in \Lambda_K$  is a cut point if and only if there is a component  $P_i$  of  $P$  and a solid torus or a thickened torus  $W$  in  $M \setminus B$  containing  $P_i$  such that  $c$  is the fixed point of a conjugate of  $\rho(\pi_1(W)) = \rho(\pi_1(P_i))$ .*

PROOF. Let  $c \in \Lambda_K$  be a cut point. By Proposition 4.11,  $c$  is stabilized by a parabolic subgroup  $H < K$ . Since  $K$  uniformizes  $(M, P)$  there is a component  $P_i$  of  $P$  such that  $H$  is conjugate to  $\rho(\pi_1(P_i))$ . Also by Proposition 4.11, there is a non peripheral simple closed curve  $\gamma \subset \partial M \setminus P$  such that  $\rho(\gamma) \in H$  (up to conjugacy). Then  $\gamma$  is homotopic in  $M$  to a curve on  $P_i$  and by the Annulus theorem [JS, IV.3.1] there is an essential annulus joining  $\gamma$  to  $P_i$ . It follows from the definition of  $B$  that there is a solid torus or a thickened torus  $W$  in  $M \setminus B$  such that  $H$  is conjugate to  $\rho(\pi_1(W))$ .

Conversely, let  $W$  be a solid torus or thickened torus in  $M \setminus B$  containing a component  $P_i$  of  $P$ . Then  $\rho(\pi_1(W))$  is a parabolic subgroup stabilizing a point  $c \in \Lambda_K$ . Let  $F$  be a component of  $W \cap \partial M$  which is disjoint from  $P$  and let  $\gamma$  be a simple closed curve on  $F$ . The curve  $\gamma$  lifts in  $\Omega_K$  to an arc  $\widehat{\gamma}$  joining  $c$  to itself. By construction,  $\widehat{\gamma} \cup c$  separates  $\Lambda_K$ . It follows that  $c$  is a cut point. ■

Let  $W_P \subset M \setminus B$  be the union of the connected components  $W_i$  of  $M \setminus B$  such that  $\rho(\pi_1(W_i))$  is a parabolic subgroup. Recall that if  $a, b \in \Lambda_K$  are not cut points then  $a\mathcal{R}b$  if they are not separated by any cut point.

**Lemma 4.15.** *A subset  $Y \subset \Lambda_K$  is the closure of a non-singleton equivalence class of  $\mathcal{R}$  if and only if there is a connected component  $W$  of  $M \setminus W_P$  such that  $Y$  is the limit set of a conjugate of  $\rho(\pi_1(W))$ .*

PROOF. We first isolate the limit sets of the fundamental groups of the components of  $M \setminus W_P$ . We denote by  $\partial_{\chi=0}M$  the union of the component of  $\partial M$  with zero Euler characteristic and we use the identification  $(\mathbb{H}^3 \cup \Omega_K)/K \approx M \setminus \partial_{\chi=0}M$  provided by the assumption that  $K$  uniformizes  $M$ . Let  $\gamma$  be a multicurve that contains a simple closed curve in each component of  $W_P \cap \partial M$  which is disjoint from  $P$ . Each component of  $\gamma$  lifts in  $\Omega_K$  to an open arc whose closure is a Jordan curve (containing a cut point by Lemma 4.14). Let  $\tilde{\gamma} \in \widehat{\mathbb{C}}$  denote the union of all the closures of the lifts of all the components of  $\gamma$ .

Each component  $\tilde{c}$  of  $\tilde{\gamma}$  is stabilized by a subgroup  $K_{\tilde{c}} < K$  isomorphic to  $\mathbb{Z}$  and bounds in  $\mathbb{H}^3 \cup \widehat{\mathbb{C}}$  a closed disc  $\tilde{D}_{\tilde{c}}$  which is invariant under the action of  $K_{\tilde{c}}$ . We denote by  $\tilde{D} = \bigcup_{\tilde{c} \in \pi_0(\tilde{\gamma})} \tilde{D}_{\tilde{c}}$  the union of all these discs. The projection  $D$  of  $\tilde{D} \cap (\mathbb{H}^3 \cup \Omega_K)/K$  to  $(\mathbb{H}^3 \cup \Omega_K)/K \approx M \setminus \partial_{\chi=0}M$  is a family of disjoint half infinite annuli joining the components of  $\gamma$  to cusps. Each connected component  $\tilde{W} \subset (\mathbb{H}^3 \cup \widehat{\mathbb{C}})$  of  $(\mathbb{H}^3 \cup \widehat{\mathbb{C}}) \setminus D$  is the universal cover of a component  $W$  of  $M \setminus D$ . On the other hand, by the uniqueness of the balanced annulus decomposition, Theorem 4.9, every component  $W$  of  $M \setminus D$  is isotopic to a component  $M \setminus \overline{W}_P$  which we also denote by  $W$ . Thus we get that, if  $\widehat{W}$  is the closure of  $\tilde{W}$ , then  $\Lambda_K \cap \widehat{W}$  is the limit set  $\Lambda_{K_W}$  of a conjugate  $K_W$  of  $\rho(\pi_1(W))$ .

Since  $\partial W \setminus P_W$  is incompressible,  $\Lambda_{K_W}$  is connected and by Lemma 4.14 it does not have any cut point. Hence  $\Lambda_{K_W}$  lies in the closure of a non-singleton equivalence class of  $\mathcal{R}$ . Given a point  $x \in \Lambda_K \setminus \Lambda_{K_W}$ , any point in  $\Lambda_K \setminus \Lambda_{K_W}$  is separated from  $x$  in  $\widehat{\mathbb{C}}$  by the closure of a component of  $\tilde{\gamma}$  and hence by a cut point in  $\Lambda_K$ . It follows that  $\Lambda_{K_W}$  is the closure of a nonsingleton equivalence class of  $\mathcal{R}$ .

By Lemma 4.14, any point in  $\Lambda_K$  that is not a cut point lie in  $\widehat{\mathbb{C}} - \overline{\tilde{\gamma}}$ . Thus we have obtained above all the nonsingleton equivalence classes of  $\mathcal{R}$ . ■

Let  $(W, P_W)$  be the compactification of a component of  $M \setminus W_P$  equipped with its induced paring. It is easy to see that the balanced annulus decomposition of  $W$  relative to  $P_W$  is  $(B \cap W) \setminus P_W$ . Thus, after Lemmas 4.14 and 4.15 it only remains to study the relation between the sets in  $V$  and  $B$  for the subgroup  $K_W$  of  $K$  uniformizing such a submanifold  $(W, P_W)$ , i.e., with connected limit set without cut points.

**Proposition 4.16.** *Let  $K$  be a geometrically finite Kleinian group uniformizing a pared manifold  $(M, P)$  with associated representation  $\rho : \pi_1(M) \rightarrow K$ . Assume that  $\Lambda_K$  is connected without cut points and let  $B$  be the balanced annulus decomposition of  $(M, P)$ . A subset  $X$  of  $\Lambda_K$  is:*

- *an inseparable cut pair if and only if there is a solid torus component  $W$  of  $M \setminus B$  such that  $X$  is the limit set of a conjugate of  $\rho(\pi_1(W))$ ,*
- *a necklace if and only if there is an  $I$ -bundle component  $W$  of  $M \setminus B$  such that  $X$  is the limit set of a conjugate of  $\rho(\pi_1(W))$ ,*
- *a maximal inseparable set if and only there is a pared acylindrical component  $W$  of  $M \setminus B$  such that  $X$  is the limit set of a conjugate of  $\rho(\pi_1(W))$ .*

To prove this proposition we use the work of Walsh [Wah] where the author explains the relation between the bumping sets of the connected components of  $\Omega_K$  and the characteristic submanifold of  $\mathbb{H}^3/K$ . First we use the separation theorem to establish the relation between the cut pairs in  $\Lambda_K$  and the bumping set of  $\Omega_K$ .

**Lemma 4.17.** *Let  $X \subset \widehat{\mathbb{C}}$  be a continuum without any cut point. A pair  $\{a, b\} \subset \Lambda_K$  is a cut pair if and only if there are at least two components  $\mathcal{O}$  and  $\mathcal{O}'$  of  $\Omega = \widehat{\mathbb{C}} \setminus X$  such that  $\{a, b\} \subset \overline{\mathcal{O}} \cap \overline{\mathcal{O}'}$ .*

PROOF. Since  $\{a, b\}$  is a cut pair, one can find two disjoint closed sets  $A$  and  $B$  of  $X \setminus \{a, b\}$  which cover  $X \setminus \{a, b\}$ . By the separation theorem [Why2, Theorem VI.3.1], there is a Jordan curve  $c$  that separates a point  $a \in A$  from a point  $b \in B$  such that  $c \cap X = \{a, b\}$ . The set  $c \setminus \{a, b\}$  has two connected components  $k$  and  $k'$ . Since  $c$  separates  $a$  from  $b$ ,  $\bar{k}$  and  $\bar{k}'$  are not homotopic in  $\Omega$  relatively to their endpoints. Since  $X$  is connected,  $\Omega$  is simply connected. It follows that  $k$  and  $k'$  do not lie in the same connected component of  $\Omega$ . The two components  $\mathcal{O}$  and  $\mathcal{O}'$  of  $\Omega$  containing  $k$  and  $k'$  respectively satisfy the conclusion. ■

Using this lemma and results of [Wah], see also [Lec, §2.3], we can study the stabilizer of a necklace.

**Lemma 4.18.** *Under the hypothesis of Proposition 4.16, a subset  $X$  of  $\Lambda_K$  is a necklace if and only if there is an  $I$ -bundle component  $W$  of  $M \setminus B$  such that  $X$  is the limit set of a conjugate of  $\rho(\pi_1(W))$ .*

PROOF. By definition, any pair of points in a necklace is a cut pair. It follows then from Lemma 4.17 that  $X$  lies in the frontiers of two components of  $\Omega_K$ . By [Wah, Theorem 3.1],  $X$  is the limit set of  $\rho(\pi_1(W))$  for an  $I$ -bundle component  $W$  of  $M \setminus B$ . Conversely, for such a component  $W$  of  $M \setminus B$  which is a pared  $I$ -bundle, every pair of points in the limit set of  $\rho(\pi_1(W))$  is a cut pair. It follows then from [PS, Corollary 18] that this limit set is a necklace. ■

Notice that this lemma implies that when  $X$  is a necklace, a pair  $\{a, b\} \subset X$  is the boundary at infinity of a lift of a component of  $B$  if and only if  $\{a, b\}$  is an inseparable cut pair.

**Lemma 4.19.** *Under the hypothesis of Proposition 4.16, a subset  $\{a, b\}$  of  $\Lambda_{K_Y}$  is an inseparable cut pair if and only if there is a solid torus component  $W$  of  $M \setminus B$  such that  $\{a, b\}$  is the limit set of a conjugate of  $\rho(\pi_1(W))$ .*

PROOF. As we mentioned before the statement, the conclusion follows from the proof of Lemma 4.18 when  $\{a, b\}$  lies in a necklace. If a cut pair does not lie in a necklace, then it is inseparable by [PS, Lemma 17] and, by [Grf, Lemme 7.3], it is stabilized by a cyclic subgroup  $K_{a,b}$  of  $K$ . By Lemma 4.17, there are two components  $\mathcal{O}$  and  $\mathcal{O}'$  of  $\Omega_K$  such that  $\{a, b\} \subset \overline{\mathcal{O}} \cap \overline{\mathcal{O}'}$ . Then  $\mathcal{O} \cup \mathcal{O}'$  is stabilized by a finite index subgroup  $K'_{a,b}$  of  $K_{a,b}$  by [Lec, Affirmation 2.5] for example. A pair of  $K'_{a,b}$ -invariant arcs  $\tilde{\gamma} \subset \mathcal{O}$  and  $\tilde{\gamma}' \subset \mathcal{O}'$  joining  $x$  to  $y$  project to a pair of homotopic closed curves  $\gamma, \gamma' \subset \partial M$  which bound an annulus  $E$  in  $M$ . It follows from Lemma 4.18 that  $E$  can not be homotoped into an  $I$ -bundle component of  $M \setminus B$ , hence  $E$  is homotopic to an annulus in  $B$  and any annulus in  $B$  that can not be homotoped into an  $I$ -bundle component of  $M \setminus B$  appears in this way. ■

**Lemma 4.20.** *Under the hypothesis of Proposition 4.16, a subset  $X$  of  $\Lambda_{K_Y}$  is a maximal inseparable set if and only there is a pared acylindrical component  $W$  of  $M \setminus B$  such that  $X$  is the limit set of a conjugate of  $\rho(\pi_1(W))$ .*

PROOF. If  $W$  is an acylindrical component of  $M \setminus B$ , it is not hard to deduce from the arguments used in the proofs of Lemmas 4.19 and 4.18 that the limit set of  $\rho(\pi_1(W))$  is a maximal inseparable set.

Let  $x \in \Lambda_{K_Y}$  and assume that for any component  $W$  of  $M \setminus B$ ,  $x$  does not lie in the limit set of any conjugate of  $\rho(\pi_1(W))$ . Let  $y$  be another point in  $\Lambda_{K_Y}$  and let  $l \in \mathbb{H}^3$  be a geodesic joining  $x$  to  $y$ . It follows from the assumption on  $x$  that  $l$  intersects essentially infinitely many lifts of  $B$ . In particular  $x$  and  $y$  are separated by a cut pair. Since this holds for any  $y \in \Lambda_K \setminus \{x\}$ ,  $x$  does not belong to an inseparable set. The conclusion follows now from the first paragraph. ■

This concludes the proof of Proposition 4.16.

Let us now recall the definition of  $\mathcal{T}_{\Lambda_K}$ . Let  $K$  be a geometrically finite Kleinian group and consider the set  $V$  of cut points, cut pairs, necklaces and maximal inseparable sets of  $\Lambda_K$ . By [Bow4, Thm 0.2] and Proposition 4.16,  $V$  is countable. Define a graph  $\mathcal{T}_{\Lambda_K}$  with vertex set  $V$  by putting an edge between two vertices  $v_1, v_2$  if  $v_1 \subset v_2$  as subsets of  $\Lambda_K$ . By Lemma 4.14 and Proposition 4.16,  $\mathcal{T}_{\Lambda_K}$  is the tree dual to the balanced annulus decomposition of the pared manifold  $(M, P)$  uniformized by  $K$ . Thus we have:

**Corollary 4.21.** *The graphs  $\mathcal{T}_{\Lambda_K}$  and  $\mathcal{T}_B$  are isomorphic.*

From now on, we will call  $\mathcal{T}_{\Lambda_K}$  the *JSJ tree* of  $K$ . Let us notice that any homeomorphism of  $\Lambda_K$  induces an isometry on  $\mathcal{T}_{\Lambda_K}$ .

4.4.4. *Quasi-isometries and JSJ splittings.* Let  $G$  be a finitely generated group quasi-isometric to a one-ended Kleinian group  $K$ . By Proposition 2.11, we may assume that  $K$  is torsion free, geometrically finite and minimally parabolic. To obtain the splitting from Theorem 4.8, we will prove that the Bowditch boundary of  $G$  is homeomorphic to  $\Lambda_K$ . Thus we get a simplicial action of  $G$  on the JSJ tree  $\mathcal{T}_{\Lambda_K}$  and a splitting of  $G$ . We will then deduce properties (1) to (5) from the results of §4.4.2, §4.4.3 and the quasi-isometric invariance of the JSJ splitting.

Since  $K$  is a geometrically finite Kleinian group, it is hyperbolic relative to its parabolic subgroups. Since it is minimally parabolic, they are isomorphic to  $\mathbb{Z}^2$ . By [DS, Theorem 1.6],  $G$  is hyperbolic relative to a collection  $P$  of subgroups each of which is quasi-isometric to  $\mathbb{Z}^2$ . It follows then from Theorem 2.3 that the peripheral subgroups of  $G$  are virtually  $\mathbb{Z}^2$ . Furthermore, by results of Osin and Drutu-Sapir the peripheral subgroups are quasi-isometrically embedded [Hru, Corollary 8.3] and, by [DS, Theorem 1.7], the quasi-isometry maps each coset of a peripheral subgroup in a neighborhood of a coset of a parabolic subgroup and conversely. Thus we get:

**Lemma 4.22.** *Let  $G$  be a one-ended finitely generated group quasi-isometric to a geometrically finite minimally parabolic Kleinian group  $K$ , then  $G$  is hyperbolic relative to virtually Abelian rank 2 groups. Furthermore the quasi-isometry preserves the peripheral structures.*

By Theorem 3.16, the quasi-isometry in Lemma 4.22 can be extended to a quasi-isometry between cusped spaces  $X_G$  and  $X_K$  which, by Proposition 3.19, induces a quasi-isometry between  $X_G$  and the convex hull of  $\Lambda_K$ . Such a quasi-isometry extends to a homeomorphism from the Bowditch boundary  $\partial_{\mathbb{P}}G$  to the limit set  $\Lambda_K$  of  $K$ . Thus we have:

**Lemma 4.23.** *Let  $G$  be a one-ended finitely generated group quasi-isometric to a geometrically finite minimally parabolic Kleinian group  $K$ , then  $G$  is relatively hyperbolic and its Bowditch boundary is homeomorphic to  $\Lambda_K$ .*

The action of  $G$  on its Bowditch boundary induces a simplicial action on the JSJ tree  $\mathcal{T}_{\Lambda_K}$  and hence a splitting, which we call the *JSJ splitting of  $G$* . This in turn fits in the more general setting of JSJ decompositions of finitely generated groups established in [GL]. The last ingredient we need to prove Theorem 4.8 is the quasi-isometric invariance of the JSJ splitting. Notice that by definition a vertex group of the JSJ splitting of  $G$  is the stabilizer of a cut point, an inseparable cut pair, a necklace or a maximal inseparable set in  $\partial_{\mathbb{P}}G$  and an edge group stabilizes a cut point or an inseparable cut pair and a necklace or a maximal inseparable set.

**Proposition 4.24** (Quasi-isometric invariance of the JSJ splitting). *Let  $G$  be a one-ended group quasi-isometric to a minimally parabolic Kleinian group  $K$  and let  $X$  be a necklace or a maximal inseparable subset of  $\partial_{\mathbb{P}}G \approx \Lambda_K$ . Let  $(G_X, \mathbb{P}_X)$ , resp.  $(K_X, \mathbb{Q}_X)$ , be the stabilizer of  $X$  in  $G$ , resp. in  $K$ , equipped with the paring induced by the stabilizers of cut points, cut pairs and parabolic points in  $X$ . Then  $(G_X, \mathbb{P}_X)$  is quasi-isometric to  $(K_X, \mathbb{Q}_X)$ .*

Using Proposition 3.20, it is easy to show that  $G_X$  is quasi-isometric to  $K_X$ , once we have established that  $X$  is their limit set.

**Fact 4.25.** *Let  $X \subset \partial_{\mathbb{P}}G$  be a necklace or a maximal inseparable set and let  $G_X < G$  be its stabilizer. Then  $X$  is the limit set of  $G_X$ .*

**PROOF.** By Proposition 4.16, we know that  $X$  corresponds to the limit set of  $K_X$ , so that it contains a dense collection of cut points and/or inseparable cut pairs. But, for each cut point or inseparable cut pair  $Y \subset X$ , there is an edge stabilizer in  $G$  which stabilizes  $Y$ , and  $X$  as well by the maximality of  $X$  as a cyclic or as an inseparable subset. Since  $G$  is one-ended, edge groups are infinite, cf. [Sta1, 4.A.6.6]. It follows that  $Y \subset \Lambda_{G_X}$ , and by density of such sets in  $X$ , that  $X \subset \Lambda_{G_X}$ . Since  $X$  is stabilized by  $G_X$ , we finally get  $X = \Lambda_{G_X}$ . ■

**PROOF OF PROPOSITION 4.24.** By Lemmas 4.22 and 4.23, there is a quasi-isometry  $\varphi : G \rightarrow K$  that preserves the parabolic structure and that defines a homeomorphism  $\partial\varphi : \partial_{\mathbb{P}}G \rightarrow \Lambda_K$ . By [Grf, Theorem 7.1],  $G_X$  and  $K_X$  are relatively quasiconvex. It follows from Fact 4.25 and Proposition 3.20 that  $\varphi(G_X)$  is at bounded distance from  $K_X$  and Fact 3.11 ensures that  $G_X$  and  $K_X$  are quasi-isometric. It remains to prove that the quasi-isometry preserves the parings.

Proposition 3.20 already ensures that the parabolic structure is preserved, so we only need to consider the stabilizers of cut pairs. Let  $P \in \mathbb{P}_X$  be the stabilizer in  $G_X$  of an inseparable cut pair  $C \subset X$ . We know that  $P$  is virtually cyclic and  $P \subset G$  is a quasi-geodesic with end points  $C$ . Then  $\varphi(P)$  is a quasi-geodesic with endpoints  $\partial\varphi(C)$ . Since  $\partial\varphi(C)$  is an inseparable cut pair, there are  $k \in K$  and  $Q \in \mathbb{Q}_X$  such that  $kQk^{-1}$  stabilizes  $\partial\varphi(C)$  and  $\partial\varphi(X)$ . Since  $Q$  is cyclic,  $kQ$  is a quasi-geodesic with endpoints  $\partial\varphi(C)$ . Now  $\varphi(P)$  and  $kQ$  are quasi-geodesics with the same endpoints, hence their distance is bounded.



This proves that  $(G_X, P_X)$  and  $(K_X, Q_X)$  are quasi-isometric as pared groups. ■

We are now ready to prove Theorem 4.8.

**PROOF OF THEOREM 4.8.** Let  $G$  be a finitely generated one-ended group quasi-isometric to a geometrically finite and minimally parabolic Kleinian group  $K$ . Lemma 4.23 provides an action of  $G$  on the JSJ tree  $\mathcal{T}_{\Lambda_K}$ . We will show that the induced JSJ splitting has Properties (1) to (5), starting with the easiest properties.

Since  $G$  is hyperbolic relatively to subgroups virtually isomorphic to  $\mathbb{Z}^2$ , Abelian subgroups are finite, virtually cyclic or virtually  $\mathbb{Z}^2$ . Since  $G$  is one-ended, edge and vertex groups are infinite, cf. [Sta1]. Property (3) follows.

A group of isometries of a Gromov hyperbolic space is Abelian only if it is elementary, i.e., its limit set is made up of at most two points. A consequence of Fact 4.25 is that Abelian vertex groups are precisely the stabilizers of cut points and inseparable cut pairs. Property (4) follows then from the definition of the JSJ tree.

Next we show Properties (1) and (2). As previously mentioned an edge group stabilizes in  $\partial_{\mathbb{P}}G$  a cut point or an inseparable cut pair and a necklace or a maximal inseparable set. If a subgroup of  $G$  stabilizes a pair of points  $\{a, b\}$ , then it is virtually cyclic. Properties (1) and (2) follow for edge groups which stabilize a pair of points.

If a subgroup of  $G$  stabilizes a single point, it is a subgroup of a peripheral subgroup of  $G$  which is virtually  $\mathbb{Z}^2$  by Lemma 4.22. If an edge group stabilizes a cut point  $\{a\}$  and a necklace or a maximal inseparable set  $X$  then it stabilizes the connected component of  $\partial_{\mathbb{P}}G \setminus \{a\}$  containing  $X$ . Properties (1) and (2) follow in that case from Lemma 4.13.

Finally Property (5) follows from Proposition 4.24 and the fact that the JSJ tree is dual to the balanced annulus decomposition (Lemma 4.14 and Proposition 4.16). ■

## 5. PRESCRIPTION OF SUBGROUPS UP TO FINITE-INDEX

We show how to construct finite-index subgroups of JSJ decompositions with prescribed vertex and edge groups.

**5.1. Finite index subgroups with prescribed peripheral subgroups.** Let  $G$  be a group. A subgroup  $H < G$  is *separable* if, for any  $g \in G \setminus H$ , there exists a finite index subgroup  $G' < G$  which contains  $H$  but not  $g$ . The group  $G$  is *residually finite* if  $\{1\}$  is separable; in other words, for any  $g \neq 1$ , there exists a finite index subgroup  $G' < G$  disjoint from  $g$ . Equivalently, for any  $g \neq 1$ , there exists a normal finite index subgroup  $G' < G$  disjoint from  $g$ . A group is *LERF* (Locally Extended Residually Finite) if any finitely generated subgroup is separable. Two groups  $G$  and  $Q$  are *commensurable* when  $G$  has a finite index subgroup isomorphic to a finite index subgroup of  $Q$ . Notice that if  $Q$  is residually finite or virtually torsion free, then  $G$  is as well.

**Definition 5.1** (Deep residually finite Dehn fillings). *A pared group  $(G, \mathbb{P})$ ,  $\mathbb{P} = \{P_1, \dots, P_n\}$ , has deep residually finite Dehn fillings if it satisfies the following property (\*):*

(\*) for each  $P_j \in \mathbb{P}$ , there exists a finite index subgroup  $P_j^\circ < P_j$  such that, whenever  $P_j^c < P_j^\circ$  is a normal finite index subgroup of  $P_j$  for each  $j$ , the quotient  $\overline{G} = G / \ll P_j^c, P_j \in \mathbb{P} \gg$  is residually finite and  $\overline{P}_j = P_j / P_j^c$  embeds in  $\overline{G}$ , where  $\ll P_j^c, P_j \in \mathbb{P} \gg$  denotes the smallest normal subgroup that contains  $\{P_j^c, P_j \in \mathbb{P}\}$ .

Let us explain the analogy with 3-manifolds. A Dehn filling of a compact 3-manifold  $M$  consists in gluing a solid torus along a toroidal boundary component of  $M$ . On the level of fundamental groups, if  $P < \pi_1(M)$  is the fundamental group of the toroidal boundary component, a Dehn filling yields a subgroup  $P^c < P$  in each  $P$  such that the fundamental group of the filled manifold is isomorphic to  $\pi_1(M) / \ll P^c, P \in \mathbb{P} \gg$ .

We will use this property to find finite index pared subgroups with prescribed parings through the following claim:

**Claim 5.2.** *Let  $(G, \{P_1, \dots, P_n\})$  be a pared group with deep residually finite Dehn fillings.*

*Then, whenever  $P_j^c < P_j^\circ$  is a normal finite index subgroup of  $P_j$  for each  $j$ , there exists a normal finite index pared subgroup  $(H, \mathbb{Q})$  of  $(G, \mathbb{P})$  such that, for any  $g \in G$  and any  $j \in \{1, \dots, n\}$ ,  $H \cap gP_jg^{-1} = gP_j^cg^{-1}$ .*

PROOF. By construction  $\overline{P}_j = P_j / P_j^c$  is finite. By property (\*),  $\overline{G}$  is residually finite and there is a morphism  $f : \overline{G} \rightarrow Q$  to a finite group  $Q$  such that  $1 \notin f(\overline{P}_j \setminus \{1\})$  for all  $j$ . Let  $H$  be the kernel of  $G \rightarrow \overline{G} \rightarrow Q$ . Then  $H$  is a normal finite index subgroup of  $G$  such that, for any  $g \in G$  and any  $j \in \{1, \dots, n\}$ ,  $H \cap gP_jg^{-1} = gP_j^cg^{-1}$ . We obtain a paring from Fact 3.15 ■

Our chief concern for the deep residually finite Dehn filling property (\*) is to Kleinian groups, but this notion could be of interest for other classes. It is inspired by the work of Wise [Wis].

**Theorem 5.3** (Residually finite Dehn filling). *A geometrically finite Kleinian group has a finite index pared subgroup  $(G, \mathbb{P})$  that has deep residually finite Dehn fillings.*

PROOF. The proof consists in combining the fact that Kleinian groups are virtually compact special, [Wis, Theorem 17.14]—this can also be proved by mapping the initial Kleinian group into a subgroup of a Kleinian group with finite covolume (see [Bro] for example) and then using [CF], [GM2] and [SaW]—with a generalization of the malnormal special quotient theorem [Wis, Lemma 15.6], see also [Ein, Theorem 2], to establish that any Kleinian group has a finite index subgroup with deep virtually compact special Dehn fillings, i.e., with property (\*) replacing “residually finite” with “virtually compact special”. The conclusion follows from the residual finiteness of compact special groups, [HW] and [Mal]. ■

The next technical lemma will allow us to avoid passing to subgroups and will simplify the arguments and notations.

**Lemma 5.4.** *Let  $(G, \mathbb{P})$  be a pared group and  $(H, \mathbb{Q})$  a finite index normal subgroup with its induced paring coming from Fact 3.15. Then  $(G, \mathbb{P})$  has deep residually finite Dehn fillings if  $(H, \mathbb{Q})$  does.*

With Theorem 5.3, this gives:

**Corollary 5.5.** *A geometrically finite Kleinian group has deep residually finite Dehn fillings.*

PROOF OF LEMMA 5.4. We use the notation coming from Fact 3.15. Assume that  $(H, \mathbb{Q})$  has deep residually finite Dehn fillings. For  $P \in \mathbb{P}$  and  $a \in T_P$ , we write  $Q_{P,a} = aPa^{-1} \cap H$ . Consider  $Q_{P,a}^\circ < Q_{P,a}$  such that, whenever  $Q_{P,a}^\circ < Q_{P,a}^\circ$  is a finite index subgroup of  $Q_{P,a}$ , the quotient  $\overline{H} = H / \ll Q_{P,a}^\circ, P \in \mathbb{P}, a \in T_P \gg$  is residually finite and  $\overline{Q_{P,a}} = Q_{P,a} / Q_{P,a}^\circ$  embeds in  $\overline{H}$ .

For every  $P \in \mathbb{P}$ , set  $P^\circ = \cap_{a \in T_P} a^{-1} Q_{P,a}^\circ a$  and note that  $P^\circ$  is a subgroup of  $H$ . Let us consider finite index subgroups  $P^c < P^\circ$ , normal in  $P$  and let  $N_G$  denote the normal closure  $\ll P^c, P \in \mathbb{P} \gg$  in  $G$ .

Fix  $P \in \mathbb{P}$  and  $a \in T_P$ . Set  $Q_{P,a}^c = aP^c a^{-1} \cap H$ : this is a finite index subgroup of  $Q_{P,a}^\circ$ . Define  $N_H$  as the normal closure of  $\{Q_{P,a}^c, P \in \mathbb{P}, a \in T_P\}$  in  $H$ . Let us prove that  $N_H = N_G = N$ . Let us first note that, for any  $P \in \mathbb{P}$ ,  $a \in T_P$  and for any  $h \in H$ ,  $hQ_{P,a}^c h^{-1} = (ha)P^c(ha)^{-1} \cap H \subset N_G$ ; this implies that  $N_H \subset N_G$ . Conversely, let  $P \in \mathbb{P}$  and  $g \in G$ . By definition of  $T_P$ , we may find  $h \in H$ ,  $a \in T_P$  and  $p \in \mathbb{P}$  such that  $g = hap$ . Therefore,  $gP^c g^{-1} = ha(pP^c p^{-1})(ha)^{-1} = hQ_{P,a}^c h^{-1} \in N_H$  since  $P^c$  is normal in  $P$ . This enables us to conclude that  $N_G = N_H = N$ .

We may now conclude that  $G/N$  is residually finite since this is the case for  $H/N$  and since  $H/N$  has finite index in  $G/N$ . Moreover, since  $Q_{P,a} / Q_{P,a}^\circ$  embeds in  $H/N$ , we may infer that  $N \cap Q_{P,a} = N_H \cap Q_{P,a} = Q_{P,a}^c$  and  $N \cap P = N \cap Q_{P,1} = Q_{P,1}^c = P^c$  so we may conclude that  $P/P^c$  embeds in  $G/N$ . ■

**5.2. Graph of groups structure with prescribed vertex groups.** We now use the previous results to “choose” the vertex and edge groups in a graph of groups.

**Proposition 5.6.** *Let  $G$  be a finitely generated group. We assume that  $G$  is the fundamental group of a finite graph of groups  $\mathcal{G} = (\Gamma, \{G_v\}, \{G_e\}, j_e : G_e \hookrightarrow G_{t(e)})$  with the following properties:*

- (i) *The set of vertices admits a partition  $V(\Gamma) = A \sqcup B$  into two types.*
  - (a) *To a vertex of type  $A$  corresponds a group  $G_v$  which admits a paring  $(G_v, \mathbb{P}_v)$  containing the adjacent edge groups and which has the deep residually finite Dehn fillings property (\*).*
  - (b) *To a vertex of type  $B$  corresponds a group  $G_v$  such that any finite index subgroup of an adjacent edge group is separable in  $G_v$ .*
- (ii) *No edge has both extremities in  $B$ .*

*Given a finite index subgroup  $H_v < G_v$  (resp.  $H_e < G_e$ ) for each vertex group  $G_v$  (resp. edge group  $G_e$ ),  $G$  contains a normal finite index subgroup  $G'$  which is the fundamental group of a finite graph of groups  $\mathcal{G}' = (\Gamma', \{G'_v\}, \{G'_e\}, G'_e \hookrightarrow G'_{t(e)})$  with the following properties.*

- (1) *Vertex groups are conjugate (within  $G$ ) to finite-index subgroups of  $H_v$ .*
- (2) *Edge groups are conjugate (within  $G$ ) to finite-index subgroups of  $H_e$ .*

If two edge-groups are commensurable in a vertex group  $G_v$ , then they are equal in  $G'_v$  (this can only happen to type B vertices).

PROOF. For each vertex subgroup  $G_v$ , we will find a finite index subgroup  $G'_v$  of  $H_v$ , normal in  $G_v$ , so that if  $e \in E$  is an edge, then  $j_e^{-1}(G'_{t(e)}) = j_{\bar{e}}^{-1}(G'_{t(\bar{e})})$ . This will allow us to construct a quotient of  $G$  in order to form a finite index subgroup of  $G$  that combines these subgroups together.

Considering instead their normal core, we may assume that the subgroups  $H_v$  and  $H_e$  are normal (and of finite index) in  $G_v$  and  $G_e$  respectively.

For  $v \in A$ , property (\*) provides us for each  $P \in \mathbb{P}_v$  a finite index subgroup  $P_v^\circ < P$  which fixes us the necessary deepness to perform controlled Dehn fillings. For  $P_v = j_e(G_e)$ ,  $v = t(e)$ , we will also write  $P_v^\circ = P_e^\circ$ .

Let  $e \in E$ ; by condition (ii) of the statement, either a single extremity is in  $A$ , and then, we may assume that it is  $t(e)$ , or both extremities are in  $A$ . In the former case, set  $K_e = H_e \cap j_e^{-1}(H_{t(e)} \cap P_e^\circ)$  and in the latter case, set  $K_e = H_e \cap j_e^{-1}(H_{t(e)} \cap P_e^\circ) \cap j_{\bar{e}}^{-1}(H_{t(\bar{e})} \cap P_{\bar{e}}^\circ)$ . In both cases  $K_e$  is a finite index subgroup of  $G_e$  contained in  $H_e$ .

Fix  $v \in B$  and let us consider an edge  $e$  with  $t(e) = v$ . The group  $L_e = j_e(K_e) \cap H_v$  has finite index in  $j_e(G_e)$ , so the assumptions on vertices in  $B$  enable us to find a normal finite index subgroup  $G_{v,e} < H_v$  in  $G_v$  such that  $G_{v,e} \cap j_e(G_e) < L_e$  is a normal finite-index subgroup of  $j_e(G_e)$ . Now, we define  $G'_v = \cap_{t(e)=v} G_{v,e}$ . This is a normal finite index subgroup of  $G_v$  such that, for any edge  $e$  with  $t(e) = v$ ,  $G'_e = j_e^{-1}(G'_v)$  is a normal finite index subgroup of  $G_e$  contained in  $j_{\bar{e}}^{-1}(P_{t(\bar{e})}^\circ)$ . We let  $G'_e = G'_e$ . We note that if two edges  $e_1$  and  $e_2$  have commensurable groups  $G_{e_1}$  and  $G_{e_2}$  in  $G_v$ , then  $G'_{e_1}$  and  $G'_{e_2}$  now coincide in  $G'_v$ .

If  $e$  is an edge for which both extremities are in  $A$ , we set  $G'_e = K_e$ .

Let us now consider  $v \in A$ . Let  $e$  be an edge with  $t(e) = v$  and let  $P_v^c = j_e(G'_e)$ . For the peripheral subgroups  $P_v$  in  $\mathbb{P}_v$  which are not edges, we consider any normal finite index subgroup  $P_v^\circ < P_v$  of  $P_v$ . The deep residually finite Dehn filling property provides us through Claim 5.2 with a normal finite index subgroup  $K_v$  such that  $K_v \cap P_v = P_v^c$  for all  $P_v \in \mathbb{P}_v$ . Set  $G'_v = K_v \cap H_v$ . We note that since  $j_e(K_e) \subset H_{t(e)}$  for every edge  $e \in E$ , we obtain for each  $v \in A$  and  $e \in t^{-1}(v)$ ,  $G'_v \cap j_e(G_e) = P_v^c = j_e(G'_e)$ .

By construction,  $j_e^{-1}(G'_{t(e)}) = G'_e = j_{\bar{e}}^{-1}(G'_{t(\bar{e})})$  holds for each edge  $e$ . We may consider the graph of groups

$$\bar{\mathcal{G}} = (\Gamma, \{\bar{G}_v\}, \{\bar{G}_e\}, \bar{G}_e \hookrightarrow \bar{G}_{t(e)})$$

where  $\bar{G}_v = G_v/G'_v$  and  $\bar{G}_e = G_e/G'_e$ . Let  $\bar{G}$  be the fundamental group of  $\bar{\mathcal{G}}$ , which is a finite graph of finite groups, hence is virtually free and residually finite.

The canonical projections  $G_v \rightarrow \bar{G}_v$  define a projection  $p : G \rightarrow \bar{G}$ . Since  $\bar{G}$  is residually finite, there is a morphism  $\varphi : \bar{G} \rightarrow F$  to a finite group  $F$  that maps a set of transversals for each  $G'_v < G_v$  to non trivial elements. Let  $G' = \ker(\varphi \circ p)$  be the kernel of  $(G \rightarrow \bar{G} \rightarrow F)$ . Then  $G' \cap G_v = G'_v$  for any vertex  $v$ . The action of  $G'$  on the Bass-Serre tree of  $G$  defines a finite graph of groups structure for which vertex groups are conjugate to some subgroup  $G'_v$  and edge groups are conjugate to some subgroup  $G'_e$ . ■

**Remark 5.7.** When  $G'_e$  is cyclic we will need that for any adjacent vertex  $v$  of type  $B$ ,  $j_e(G'_e)$  is primitive in  $G'_v$ , i.e., for  $g^n \in j_e(G'_e)$  with  $g \in G'_v$  and  $n \geq 1$  then  $g \in j_e(G'_e)$ . We will have only two cases to consider, when  $G'_v$  is cyclic and when  $G'_v$  is isomorphic to  $\mathbb{Z}^2$  and all adjacent edge groups are equal. In both cases it is easy to replace  $G'_v$  by a finite index subgroup such that any adjacent edge group is primitive without changing the edge groups.

We draw the following application to 3-manifolds:

**Proposition 5.8.** *Let  $M$  be a compact irreducible 3-manifold. There is a finite cover  $N \rightarrow M$  such that  $N$  is orientable, and if  $N$  admits a non trivial torus decomposition, then each Seifert piece is the product of  $S^1$  with an orientable compact surface.*

PROOF. Taking a degree two cover, we may assume that  $M$  is orientable. When  $M$  is a Sol-manifold, the result can be found in [Sco3, Theorem 4.17]. If  $M$  is a graph-manifold, it follows for instance from [KaL1, Lemma 2.1]. Otherwise, the characteristic torus decomposition provides us with a graph of groups structure such that each piece is either hyperbolic or a graph-manifold, see §4.3. For the latter pieces, there are finite covers so that Seifert pieces are trivial  $S^1$ -bundles over orientable surfaces [Hem1, Theorem 12.2]. By [Ham] and Corollary 5.5, the hypotheses of Proposition 5.6 are fulfilled, and it produces a normal finite index subgroup  $G'$  so that each non-hyperbolic piece corresponds to a product  $\mathbb{Z} \times \pi_1(S)$  where  $S$  is an orientable surface. The corresponding cover satisfies the conclusions of the proposition. ■

## 6. QUASI-ISOMETRIC RIGIDITY OF 3-MANIFOLDS GROUPS

In this section, we gather our previous results to establish Theorems 1.2, 1.4 and 1.3, starting with the latter since it is used in the two other proofs.

**6.1. Quotients by finite subgroups vs finite index subgroups.** Our main use for separability is the following proposition, see [Hai1, Prop. 7.2] for a proof.

**Proposition 6.1.** *Let  $A' < A < G$  be groups with  $[A : A'] < \infty$  and  $A'$  separable in  $G$ . Then there exist subgroups  $A''$  and  $H$  with the following properties:*

- (1)  $H$  is a normal subgroup of finite index in  $G$ ;
- (2)  $A'' = H \cap A'$  is a normal subgroup of finite index in  $A$ ;
- (3) for all  $g \in G$ ,  $(gAg^{-1}) \cap H = gA''g^{-1}$ .

Our goal here is to present some results to conclude that groups quasi-isometric to 3-manifolds groups are residually finite, cf. Theorem 1.3.

**Lemma 6.2.** *Let  $G$  be a finitely generated group and  $p : G \rightarrow Q$  a morphism with finite kernel. Assume that  $Q$  has a graph of groups structure  $(\Gamma, \{Q_v\}, \{Q_e\}, i_e : Q_e \hookrightarrow Q_{t(e)})$  with the following properties.*

- (1) every finite index subgroup of an edge group  $Q_e$  is separable in  $Q$ ;
- (2) for every vertex  $v$ ,  $p^{-1}(Q_v)$  is residually finite.

Then  $G$  and  $Q$  are commensurable.

PROOF. We will produce a morphism  $q : G \rightarrow \overline{G}$  onto a virtually free group which is injective on  $\ker p$ , and use the separability of free groups to conclude. The construction of the projection  $q$  will result from taking compatible quotients of the vertex groups of  $G$  onto finite subgroups.

The action of  $Q$  on its Bass-Serre tree yields through  $p$  an action of  $G$ , inducing a graph of groups structure

$$(\Gamma, \{G_v = p^{-1}(Q_v)\}, \{G_e = p^{-1}(Q_e)\}, j_e : p^{-1}(Q_e) \hookrightarrow p^{-1}(Q_{t(e)}))$$

for  $G$  such that  $i_e \circ p = p \circ j_e$  for every edge (this can be deduced for example from [Ser, Theorems 12 and 13]).

By assumption each vertex group  $G_v = p^{-1}(Q_v)$  is residually finite, hence there is a normal finite index subgroup  $G'_v$  of  $G_v$  such that  $G'_v \cap \ker p = \{1\}$ .

For any edge  $e$ , set  $G'_e = j_e^{-1}(G'_{t(e)}) \cap j_{\bar{e}}^{-1}(G'_{t(\bar{e})})$ . By construction, the restriction  $p : G'_e \rightarrow Q_e$  is injective and the group  $Q'_e = p(G'_e)$  is a normal finite index subgroup of  $p(G_e) = Q_e$ . By assumption,  $Q'_e$  is separable and by Proposition 6.1, there is a normal finite index subgroup  $N_e$  in  $Q$  such that  $N_e \cap Q_e \subset Q'_e$ .

Let  $Q' = \cap_e N_e$ , which has finite index in  $Q$  since  $N_e$  has finite index in  $Q$  for any edge  $e$  and  $\Gamma$  is finite. For each vertex  $v$ , set  $H_v = p^{-1}(Q') \cap G'_v$ . This subgroup  $H_v$  is normal and has finite index in  $G_v$  since the same holds for  $G'_v$  in  $G_v$  and for  $Q'$  in  $Q$  and since  $\ker p$  is finite; furthermore, we have  $H_v \cap \ker p \subset G'_v \cap \ker p = \{1\}$ .

Let us fix an edge  $e$ . Since  $p$  is injective on  $j_e^{-1}(G'_{t(e)}) \cup j_{\bar{e}}^{-1}(G'_{t(\bar{e})})$  and  $Q' \cap p(G_e) \subset Q'_e$ , we have

$$j_e^{-1}(H_{t(e)}) = (p \circ j_e)^{-1}(Q') \cap G'_e = (p \circ j_{\bar{e}})^{-1}(Q') \cap G'_e = j_{\bar{e}}^{-1}(H_{t(\bar{e})})$$

and this subgroup that we name  $H_e$  is normal in  $G_e$ .

This implies that we may define a graph of groups

$$\overline{\mathcal{G}} = (\Gamma, \{\overline{G}_v\}, \{\overline{G}_e\}, \overline{G}_e \hookrightarrow \overline{G}_{t(e)})$$

where  $\overline{G}_v = G_v/H_v$  and  $\overline{G}_e = G_e/H_e$ . Let  $\overline{G}$  be the fundamental group of  $\overline{\mathcal{G}}$ , which is a finite graph of finite groups, hence  $\overline{G}$  is virtually free [ScW, Thm. 7.3] and residually finite.

The canonical projections  $G_v \rightarrow \overline{G}_v$  define a projection  $q : G \rightarrow \overline{G}$  and  $q$  is injective on  $\ker p$ . Since  $\overline{G}$  is residually finite, there is a morphism  $r : \overline{G} \rightarrow F$  to a finite group which is injective on  $q(\ker p)$ . The kernel  $H = \ker(r \circ q)$  is a finite index subgroup of  $G$  which avoids  $\ker p \setminus \{1\}$ , so  $H$  embeds in  $Q$  as a finite index subgroup. ■

Let  $G$  be a group. A *hierarchy of length 0* for  $G$  is the graph of groups with trivial and with the single vertex group  $G$ . Let  $n > 0$ , a *hierarchy of length at most  $n$*  for  $G$  is a non-trivial finite graph of groups structure for  $G$  together with a hierarchy of length at most  $n - 1$  for each vertex group. The hierarchy has *length  $n$*  if it has length at most  $n$  and at least one vertex group has a hierarchy of length  $n - 1$ .

Let  $G_v$  be a vertex group of the graph of groups structure of  $G$ . We say that  $G_v$  is *at level 1* of the hierarchy. The groups at level 1 for the hierarchy of  $G_v$  are *at level 2* of the hierarchy for  $G$  and so on. The vertex groups with a hierarchy of length 0 are called the *terminal groups* of the hierarchy.

Using Lemma 6.2 inductively on a complete hierarchy, we get the following:

**Lemma 6.3.** *Let  $G$  be a group and  $p : G \rightarrow Q$  a morphism with finite kernel to a group  $Q$  which has a hierarchy of length  $n$ . A group  $H$  at level at most  $n - 1$  in the hierarchy of  $Q$  comes with a graph of groups structure  $(\Gamma, \{H_v\}, \{H_e\}, H_e \hookrightarrow H_{t(e)})$ . Assume that for any such group  $H$  (including  $Q$  itself) every finite index subgroup of an edge group  $H_e$  is separable in  $H$ . Assume furthermore that for any terminal group  $T$  in the hierarchy for  $Q$ ,  $p^{-1}(T)$  is residually finite. If  $Q$  is residually finite, then  $G$  and  $Q$  are commensurable.*

PROOF. From the hierarchy for  $Q$  the morphism  $p$  induces a hierarchy of length  $n$  for  $G$  with the property that if  $G_v$  is a group at level  $k$  of the hierarchy for  $G$  then  $G_v = p^{-1}(Q_v)$  for some group  $Q_v$  at level  $k$  of the hierarchy for  $Q$ .

We are going to prove Lemma 6.3 with a finite recurrence on the level in reverse order. Our induction hypothesis is:

$(P(k))$  Any group  $G_v$  at level  $k$  of the hierarchy for  $G$  is residually finite and commensurable to  $Q_v = p(G_v)$ .

$P(n)$  is satisfied by assumption.

Assume  $P(k)$  holds for a given  $k$  ( $1 \leq k \leq n$ ) and let us show that  $P(k - 1)$  holds. Let  $G_v$  be a group at level  $k - 1$  of the hierarchy for  $G$ , by construction,  $Q_v = p(G_v)$  is a group at level  $k - 1$  of the hierarchy for  $Q$ . Since  $P(k)$  holds, by the assumption made on the hierarchy for  $Q$ ,  $G_v$  and  $Q_v$  satisfy the assumption of Lemma 6.2. It follows that  $G_v$  is commensurable to  $Q_v = p(G_v)$ . Since  $Q$  is residually finite, the same is true for  $Q_v$ , hence for  $G_v$ . ■

We may now prove Theorem 1.3.

**Theorem 1.3.** *Let  $G$  be a finitely generated group and  $p : G \rightarrow Q$  a morphism with finite kernel. If  $Q$  has a finite index subgroup isomorphic to the fundamental group of a compact 2- or 3-manifold  $M$  then  $G$  is commensurable to  $Q$ .*

PROOF. Up to taking a finite index subgroup of  $G$ , we may assume that  $Q$  is isomorphic to the fundamental group of  $M$  and that  $M$  is orientable. If  $G$  is finite,  $G$  and  $Q$  are commensurable to the trivial group, so we may assume that  $G$  and  $Q$  are infinite.

If  $M$  is a surface, it is easy to find a hierarchy for  $Q$  with trivial terminal groups (see [ScW] for example). According to [Sco2, Sco4, Theorems 3.3],  $Q$  is LERF and the conclusion follows from Lemma 6.3.

If  $M$  is a compact irreducible 3-manifold, we know from Agol [Ago2] that  $M$  is virtually Haken. So, up to taking a finite index subgroup of  $G$ , we may assume that  $M$  is Haken.

Let  $T$  be the characteristic torus decomposition of  $M$ . If  $T$  is empty, then  $M$  is Seifert or atoroidal (hence hyperbolic by Theorem 2.9). Fundamental groups of Seifert and hyperbolic manifolds are LERF by [Sco2, Sco4, Theorems 4.1] for Seifert manifolds and [Ago2, Cor.9.4] for hyperbolic manifold (see also [Wis, Corollary 17.4]). The Haken hierarchy [Hak1, Hak2] in  $M$  provides us with a hierarchy for  $Q$  with trivial terminal groups. Again we conclude with Lemma 6.3.

If  $T \neq \emptyset$ , which also includes the case of *Sol* manifolds, then it induces a graph of groups structure  $(\Delta, \{Q_v\}, \{Q_e\}, Q_e \hookrightarrow Q_{t(e)})$  for  $Q$  with edge groups  $Q_e$  isomorphic to  $\mathbb{Z}^2$  and vertex groups  $G_v$  isomorphic to fundamental groups of Seifert or hyperbolic manifold  $M_v$ . By [Ham, Corollary 17.4], any subgroup of an edge group is separable in  $Q$ . By the hyperbolization theorem, every vertex manifold  $M_v$  is geometric, in particular  $Q_v$  is linear and hence residually finite by Malcev's theorem [Mal]. We have already proved Theorem 1.3 for Seifert and hyperbolic manifolds so we know that  $p^{-1}(Q_v)$  is comensurable to  $Q_v$ . Thus  $p^{-1}(Q_v)$  is residually finite and the conclusion follows from Lemma 6.2.

If  $M$  contains an essential sphere,  $Q$  has a graph of groups structure with trivial edge groups and vertex groups isomorphic to fundamental groups of compact irreducible 3-manifolds. It follows from the previous paragraphs that Theorem 1.3 holds for compact irreducible 3-manifolds. Since fundamental groups of compact 3-manifolds are residually finite [Hem2, Corollary 1.2], the conclusion follows again from Lemma 6.2. ■

**6.2. Acylindrical hyperbolic manifolds and  $I$ -bundles.** The goal of this section is to establish the quasi-isometric rigidity of non-Abelian vertex groups of a JSJ splitting in the sense of §4.4. More concretely we are going to show the quasi-isometric rigidity of pared fundamental groups of pared  $I$ -bundles and of acylindrical hyperbolic pared manifolds.

**Lemma 6.4.** *Let  $(G, \mathbb{P})$  be a pared group quasi-isometric to the pared fundamental group of a pared  $I$ -bundle. Then  $(G, \mathbb{P})$  has a finite index pared subgroup which is the pared fundamental group of a pared  $I$ -bundle.*

PROOF. Let  $(W, \mathbb{P}_W) = (F \times I, \partial F \times I)$  be a pared  $I$ -bundle whose pared fundamental group  $(K, \mathbb{P}_K)$  is quasi-isometric to  $(G, \mathbb{P}_G)$ . If  $\partial F = \emptyset$ , the conclusion follows from Theorem 2.5. Otherwise  $K$  is a free group and by Theorem 2.1,  $G$  has a free finite index subgroup  $H$ . The induced paring  $(H, \mathbb{P}_H)$  given by Fact 3.15 is thus quasi-isometric to  $(W, \mathbb{P}_W)$ . Considering a finite volume hyperbolic structure on the interior of  $F$  and applying Proposition 3.19, we see that  $\partial\mathcal{Cus}(K, \mathbb{P}_K)$  is homeomorphic to  $S^1$ . By Theorem 3.16  $\partial\mathcal{Cus}(H, \mathbb{P}_H)$  is also homeomorphic to a circle and by Proposition 3.22 and [Ota1, Theorem 2],  $(H, \mathbb{P}_H)$  is the pared fundamental group of a pared  $I$ -bundle. ■

To prove the quasi-isometric rigidity of pared fundamental groups of acylindrical hyperbolic manifold, we first establish the rigidity of their action on their limit set.

**Theorem 6.5.** *If a finitely generated group  $G$  acts minimally as a geometrically finite convergence group by quasi-Möbius mappings on the limit set of a geometrically finite Kleinian group  $K$  (with infinite covolume) whose convex core has totally geodesic boundary, then  $G$  is commensurable to  $K$ .*

We start with a preliminary rigidity result concerning quasi-Möbius mappings. A *Schottky set* is the complement of a family of at least three pairwise disjoint open round discs of  $\widehat{\mathbb{C}}$ .

**Theorem 6.6** (Bonk, Kleiner and Merenkov [BKM]). *Any quasi-Möbius selfmap of a Schottky set of measure zero is the restriction of a Möbius transformation.*

The proof of Theorem 6.5 is essentially the same as Corollary 3.9 in [Hai2].



PROOF OF THEOREM 6.5. Let  $\Lambda_K$  be the limit set of  $K$ , since the convex core of  $K$  has totally geodesic boundary,  $\Lambda_K$  is a Schottky set and has measure 0 by [Ahl]. Let  $F$  be the kernel of the action of  $G$  on  $\Lambda_K$  and  $G' = G/F$ . Since  $G$  acts properly discontinuously on distinct triples in  $\Lambda_K$ ,  $F$  is finite. Let  $G_M$  denote the set of quasi-Möbius selfhomeomorphisms of  $\Lambda_K$ . Note that  $G_M$  contains  $K \cup G'$ .

According to Theorem 6.6, the action of  $G_M$  extends to an action of Möbius transformations on  $\widehat{\mathbb{C}}$ . It is clearly discrete since any sequence which tends uniformly to the identity will have to eventually stabilize at least three circles, implying that such a sequence is eventually the identity. Since the limit sets of  $K$  and  $G_M$  coincide and since  $K$  is geometrically finite,  $K$  has finite index in  $G_M$ , cf. [SS, Theorem 1]. Since the action of  $G'$  on  $\Lambda_K$  is minimal,  $\Lambda_K$  is the limit set of  $G'$ . By assumption this action is geometrically finite, hence again by [SS, Theorem 1],  $G'$  has finite index in  $G_M$  and  $G'$  and  $K$  are commensurable. The conclusion follows from Theorem 1.3. ■

We generalize Theorem 6.5 as follows:

**Theorem 6.7** (pared quasi-isometric rigidity). *A pared group  $(G, \mathbb{P}_G)$  is quasi-isometric to an acylindrical hyperbolic pared 3-manifold  $(M, P_M)$  if and only if there is a compact hyperbolic pared 3-manifold  $(N, P_N)$  whose pared fundamental group is isomorphic to a finite index pared subgroup of  $(G, \mathbb{P}_G)$ . Moreover,  $G$  is commensurable to  $\pi_1(M)$ .*

Notice that the compact and finite volume cases follow from [CC] and [Sch] respectively and that the case of an acylindrical pared free group has been established by Otal [Ota1].

PROOF OF THEOREM 6.7. Let  $(G, \mathbb{P}_G)$  be a pared group quasi-isometric to the pared fundamental group of an acylindrical pared hyperbolic manifold  $(M, P)$ . By Theorem 2.10, there is a geometrically finite Kleinian group  $K$  whose convex core is homeomorphic to  $M \setminus P$  and has totally geodesic boundary. As previously mentioned, when  $\mathbb{H}^3/K$  has finite volume, the conclusion follows from [CC] and [Sch]. So let us assume that the volume of  $\mathbb{H}^3/K$  is infinite. Let  $\mathbb{P}_K$  be the paring of  $K$  given by its parabolic subgroups. By assumption the image of any coset of an element of  $\mathbb{P}_G$  is at bounded Hausdorff distance from a coset of an element of  $\mathbb{P}_K$  and conversely. By Theorem 3.16, the quasi-isometry between  $(G, \mathbb{P}_G)$  and  $(K, \mathbb{P}_K)$  extends to a quasi-isometry between cusped spaces  $\mathcal{Cus}(G, \mathbb{P}_G)$  and  $\mathcal{Cus}(K, \mathbb{P}_K)$ . It follows that  $\mathcal{Cus}(G, \mathbb{P}_G)$  is hyperbolic and that  $G$  is hyperbolic relative to  $\mathbb{P}_G$ . By Proposition 3.19 the quasi-isometry between cusped spaces  $\mathcal{Cus}(G, \mathbb{P}_G)$  and  $\mathcal{Cus}(K, \mathbb{P}_K)$  induces a quasi-isometry between  $\mathcal{Cus}(G, \mathbb{P}_G)$  and the convex hull of  $\Lambda_K$  which extends to a homeomorphism  $\Phi : \partial_{\mathbb{P}_G} G = \partial \mathcal{Cus}(G, \mathbb{P}_G) \rightarrow \Lambda_K$ . Since  $G$  is relatively hyperbolic, its action on  $\partial_{\mathbb{P}_G} G$  (and hence on  $\Lambda_K$ ) is minimal and a geometrically finite convergence action. By Theorem 6.5, there is a finite index subgroup  $Q$  of  $G$  isomorphic to a finite index subgroup  $H$  of  $K$ . Let  $\mathbb{P}_Q$  and  $\mathbb{P}_H$  be the induced parings provided by Fact 3.15.

It remains to show that the isomorphism  $\varphi : Q \rightarrow H$  preserves the parings. Let us first notice that, by construction, the map induced by  $\varphi$  on  $\partial \mathcal{Cus}(Q, \mathbb{P}_Q)$  is the restriction of  $\Phi$ . Hence  $\varphi$  maps the stabilizer in  $Q$  of a point  $x \in \partial \mathcal{Cus}(G, \mathbb{P}_G) = \partial \mathcal{Cus}(Q, \mathbb{P}_Q)$  to the stabilizer in  $H$  of  $\Phi(x) \subset \Lambda_K = \Lambda_H$ . By construction, the stabilizer in  $G$  of a point  $x \in \partial \mathcal{Cus}(G, \mathbb{P}_G)$  is either trivial or conjugate to a subgroup in  $\mathbb{P}_G$  and is non-trivial only if the stabilizer in  $K$  of  $\Phi(x)$  is conjugate to a subgroup in  $\mathbb{P}_K$ . The conclusion follows. ■

**6.3. Quasi-isometric rigidity of Kleinian groups.** This section is devoted to the proof of Theorem 1.4. We apply the previous sections to build a Kleinian group from a quasi-isometry  $G \rightarrow K$ . Theorems 4.8 and 6.7 and Lemma 6.4 produce a graph of groups structure for  $G$  with virtually Kleinian vertex groups. Then we use Proposition 5.6 to get a finite index subgroup which can be obtained by gluing fundamental groups of compact 3-manifolds along their boundaries. This first leads us to prove the quasi-isometric rigidity of one-ended Kleinian groups:

**Theorem 6.8.** *Let  $G$  be a one-ended group quasi-isometric to a minimally parabolic geometrically finite Kleinian group. Then  $G$  has a finite index subgroup isomorphic to a one-ended geometrically finite Kleinian group.*

Combining this theorem with Theorem 4.3 and Proposition 4.2, we obtain the quasi-isometric rigidity of geometrically finite Kleinian groups:

**Theorem 6.9.** *Let  $G$  be a group quasi-isometric to a minimally parabolic geometrically finite Kleinian group. Then  $G$  has a finite index subgroup isomorphic to a geometrically finite Kleinian group.*

Finally, Theorem 1.4 is a consequence of the latter together with Proposition 2.11. Now it remains to prove Theorem 6.8.

**PROOF OF THEOREM 6.8.** Let  $G$  be a one-ended group quasi-isometric to a minimally parabolic geometrically finite Kleinian group. By Theorems 4.8 and 6.7 and Lemma 6.4,  $G$  has a graph of groups structure  $(\Gamma, \{G_v\}, \{G_e\}, G_e \hookrightarrow G_{t(e)})$  with the following properties:

- (1) vertex groups are of essentially two types:
  - (a) pared groups with paring containing the adjacent edges and with a pared finite index subgroup isomorphic to a pared Kleinian group;
  - (b) virtually Abelian groups of rank at most 2;
- (2) virtually Abelian vertex groups are not adjacent;
- (3) edge groups are virtually cyclic, and edge groups incident to an Abelian vertex group are commensurable (by Lemma 4.13).

By Theorem 5.3 type (a) vertices here are also type A for the definition of Proposition 5.6. Since Abelian groups are LERF (their subgroups are normal), type (b) vertices are also type B for the definition of Proposition 5.6. For each type (a) vertex  $v$  we pick a finite index normal pared Kleinian subgroup  $H_v < G_v$  and for each type (b) vertex  $w$  we pick a torsion free Abelian subgroup  $H_w < G_w$ . By Proposition 5.6,  $G$  has a finite index subgroup  $G'$  which is the fundamental group of a finite graph of groups  $\mathcal{G}' = (\Gamma', \{G'_v\}, \{G'_e\}, G'_e \hookrightarrow G'_{t(e)})$  such that any vertex group is conjugate to a finite index subgroup of  $H_v$ . In particular  $\mathcal{G}' = (\Gamma', \{G'_v\}, \{G'_e\}, G'_e \hookrightarrow G'_{t(e)})$  has the following properties:

- (1) Vertex groups are of essentially two types:
  - (a) pared geometrically finite Kleinian groups with paring containing the adjacent edges;
  - (b) Abelian groups isomorphic to  $\mathbb{Z}$  or  $\mathbb{Z}^2$ .
- (2) Edge groups are cyclic, and edge groups incident to an Abelian vertex group all coincide.

Notice that according to Remark 5.7 and the definition of pared geometrically finite Kleinian groups, we may assume that edge groups are primitive.

This graph of groups structure for  $G'$  enables us to build a Kleinian group. To each vertex  $v$  we associate a compact 3-manifold  $M_v$  with fundamental group  $G'_v$ .

If  $G'_v$  is isomorphic to  $\mathbb{Z}$ ,  $M_v$  is a solid torus and for each adjacent edge  $e$ , we pick an annulus  $A_e$  on the boundary of  $M_v$  such that the map  $i_* : \pi_1(A_e) \rightarrow \pi_1(M_v)$  induced by the inclusion is also the map  $G'_e \hookrightarrow G'_v$  defined by the graph of groups  $\mathcal{G}'$ . Since all those edge groups coincide and are primitive, we can choose the annuli corresponding to different edges to be embedded, disjoint and parallel.

If  $G'_v$  is isomorphic to  $\mathbb{Z}^2$ ,  $M_v$  is a thickened torus  $\mathbb{T} \times I$ . For each adjacent edge  $e$ , we pick an annulus  $A_e \subset \mathbb{T} \times \{0\}$  such that  $i_* : \pi_1(A_e) \rightarrow \pi_1(M_v)$  corresponds to  $G'_e \hookrightarrow G'_v$ . Again we can choose the annuli corresponding to different edges to be embedded disjoint and parallel.

Otherwise,  $G'_v$  is isomorphic to a geometrically finite Kleinian group  $K_v$ ,  $M_v$  is the compact manifold whose interior is homeomorphic to  $\mathbb{H}^3/K_v$  and the incident edge groups define a paring on  $\partial M_v$  corresponding to the parabolic subgroups of  $K_v$ . In particular to each adjacent vertex  $e$  is associated an annulus  $A_e \subset \partial M_v$ .

Given an edge  $e = (v, v')$ , we glue  $M_v$  and  $M_{v'}$  together along the annuli  $A_e \subset M_v$  and  $A'_e \subset M_{v'}$ . The manifold thus produced does not depend on the map chosen to identify the annuli (up to homeomorphism). Doing this gluing for each edge, we get a compact 3-manifold  $M$  whose fundamental group is  $G'$  [ScW].

By construction,  $M$  is irreducible. We just need to show that  $M$  is atoroidal to conclude with the hyperbolization theorem.

Since  $A$  is minimal (property (iii) of Theorem 4.9), every torus component of  $\partial M$  is contained in the boundary of a component of  $M \setminus A$ . As was already mentioned,  $Q = \pi_1(M)$  is hyperbolic relative to its rank 2 Abelian subgroups. By [DS],  $G$  is relatively hyperbolic and the rank 2 Abelian subgroups of  $Q$  are mapped at bounded Hausdorff distance from the peripheral subgroups of  $G$  by the quasi-isometry  $G \rightarrow Q$ . It follows that any rank 2 Abelian subgroup of  $G$  is a subgroup of a conjugate of a vertex group of  $\mathcal{G}$  hence that every rank 2 Abelian subgroup of  $G'$  is a subgroup of a conjugate of a vertex group of  $\mathcal{G}'$ . By construction  $M_v$  is atoroidal for every vertex  $v$ , hence  $M$  is atoroidal. The conclusion follows from the hyperbolization theorem. ■

**6.4. Quasi-isometric rigidity of 3-manifold groups.** We may now combine all our previous results together to deduce Theorem 1.2.

**Theorem 6.10.** *Let  $G$  be a group quasi-isometric to the fundamental group of a compact 3-manifold  $M$ . Then  $G$  has a finite index subgroup isomorphic to the fundamental group of a compact 3-manifold.*

**PROOF.** Let us first assume that  $G$  is one-ended. It follows that the fundamental group of  $M$  is also one-ended so that  $M$  is irreducible and  $\partial$ -irreducible, cf. §2.2. Moreover, without loss of generality, we may assume that the manifold is orientable. Then by Theorem 4.5,  $G$  has a graph of groups structure  $\mathcal{G} = (\Gamma, \{G_v\}, \{G_e\}, G_e \hookrightarrow G_{t(e)})$  such that each vertex group  $G_v$  is quasi-isometric to the fundamental group  $Q_v$  of a compact 3-manifold which is either

hyperbolic or has zero Euler characteristic. Furthermore the quasi-isometry can be chosen to map the incident edge groups to conjugates of fundamental groups of boundary components. In particular, edge groups are virtually rank 2 Abelian by Theorem 2.3.

By Theorems 4.5 and 6.9, for each vertex  $v$ ,  $G_v$  has a finite index subgroup  $H_v$  which is the fundamental group of a compact 3-manifold  $M_v$  which is either hyperbolic or has zero Euler characteristic. Furthermore, for each adjacent edge  $e = (v, w)$ ,  $H_v \cap G_e$  is conjugate to the fundamental group of a boundary components of  $M_v$ .

By Theorem 5.3, vertices  $v$  with  $M_v$  hyperbolic are type A for Proposition 5.6. Since edge groups are Abelian, it follows from [Ham] that vertices  $v$  such that  $M_v$  has zero Euler characteristic are type B. By construction of the Euler characteristic decomposition, two type B vertices can not be adjacent.

Hence  $\mathcal{G}$  satisfies the hypothesis of Proposition 5.6, it follows that  $G$  has finite index subgroup  $G'$  of  $G$  such that for any vertex  $v$ ,  $G'_v \cap G_v$  is a normal finite index subgroup of  $H_v$ . The subgroup  $G'_v$  is the fundamental group of a covering  $M'_v$  of  $M_v$ . The subgroup  $G'$  inherits from  $\mathcal{G}$  a graph of groups structure  $\mathcal{G}' = (\Gamma', \{G'_v\}, \{G'_e\}, G'_e \hookrightarrow G'_{t(e)})$  such that any vertex group  $G'_v$  is the fundamental group of a compact 3-manifold  $M'_v$  and incident edge groups are conjugate to fundamental groups of boundary components. For each edge  $e = (v, w)$  of  $\Gamma'$  we glue  $M'_v$  to  $M'_w$  along the components of their boundaries corresponding to  $e$ . This produces a compact 3-manifold whose fundamental group is  $G'$  and the conclusion follows.

When  $G$  is two-ended, then it is virtually cyclic and hence has a subgroup isomorphic to the fundamental group of a solid torus.

When  $G$  has infinitely many ends, the conclusion follows from the one-ended case together with Proposition 4.2 and Theorem 4.3. ■

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